An analysis of objective inflation expectations and inflation risk premia*

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Abstract

We propose a factor model to measure expected inflation and the inflation risk premium at different maturities that leverages two sets of market instruments, inflation swaps and inflation caps and floors. The model features time-varying long-term average inflation and variable inflation volatility and exploits the information contained in survey-based inflation forecasts to anchor the objective measures of expected inflation. Medium-term expected inflation was close to the ECB's "below, but close to" inflation aim of 2% from 2010 to 2014, it has since declined to a low in March 2020 and increased significantly in the second half of 2021, up by more than 2% from September 2021. The inflation risk premium, positive until 2014, has been negative since 2015 and reached a minimum after the outbreak of the pandemic to return to values close to zero in autumn 2021. The probability of inflation being negative over a 3-year horizon peaked above 50% in late 2014 and early 2020. The probability of exceeding the ECB's inflation aim has always been lower than 40%, with the exception of the 2011-12 period, until September 2021, after which it rose to very high values, exceeding 80% in December 2021.

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1 Introduction

Since the aftermath of the sovereign debt crisis, euro area inflation and expected inflation have been drifting downwards, raising concerns among market analysts and policymakers about the anchoring of inflation expectations to the European Central Bank's (ECB) target – see Corsello et al. (2021) for a brief review. The downward trend intensified, first in 2014 and, subsequently, with the outbreak of the Covid-19 pandemic in 2020. On the contrary, since the second half of 2021, we have been observing a fast and steep rise of consumer prices and inflation expectations. Since expected inflation plays such an important role in monetary policy decisions, a timely and reliable estimate of it is essential to define the monetary policy stance and investors' decisions on portfolio allocations.

Two sets of variables are tipically used to infer the unobserved value of expected inflation and the inflation risk premia. The first contains information on inflation implied in derivative instruments traded daily on financial markets, such as bonds, index-linked bonds, inflation swaps and inflation caps and floors, and is used to calculate what is known as breakeven inflation; this measure, being priced in traded assets, refers to a representative risk-neutral investor and is made up of a component defined as objective expected inflation and a component that rewards the uncertainty borne by the investor, the inflation risk premium. The second set of variables is obtained from analysts' surveys on expected inflation over different horizons, conducted on a monthly or quarterly basis by specialized agencies or central banks, and provides an objective measure of expected inflation, net of the risk premium component. Often the two measures not only differ due to the wedge imposed by the presence of the risk premium, but also go in different directions.[1] Therefore one may wonder which of the two measures gives the correct signal which is crucial for both investors and monetary authorities.

The aim of this paper is to retrieve reliable estimates of objective expected inflation at high frequency exploiting the information content of both types of variables.

Current models of expected inflation based on financial derivatives prices have some important drawbacks. First, they assume pricing factors that have a constant mean and, therefore, are unable to capture regime changes at low frequencies. Second, they assume a constant volatility of inflation but we know that prices are related to volatility: a higher variance of the underlying increases the prices of options as caps and floors; in general, only at-the-money options are fairly priced with the Black and Scholes model that famously assumes constant volatility, while stochastic volatility offers a very flexible and promising description of option prices. Empirical evidence tells us that, similar to interest rates, low inflation volatility is associated with a low inflation risk premium and, in turn, with low expected inflation, while

[1] See Cecchetti et al. (2021) for a discussion of the measures of inflation expectations derived from the prices of financial instruments indexed to inflation and from the surveys of professional forecasters.

high volatility is associated with high levels.[2] We model expected inflation with the aim of overcoming these drawbacks.

This paper tackles the issue of decomposing breakeven inflation into the objective and risk premium component by using both derivatives instruments and analysts' surveys. We based on a setup developed in the seminal contributions of Bansal and Yaron (2004) and Fleckenstein et al. (2017) that nests the contribution of the Heston (1993) stochastic volatility option pricing model. We use daily data of inflation swaps, that provide a direct measure of average breakeven inflation, and inflation caps and floors that provide information on the entire distribution of risk-neutral expectations of future inflation. The price of any derivative contract is simultaneously a function of both the aggregate market view of the objective probability distribution of the underlying and the aggregate market risk attitude, but to those who observe the market price, the two components seem inextricable. In our paper, we complement the information coming from the markets with survey-based measures of inflation expectations, using quarterly data of the survey of professional forecaster (SPF) conducted by the ECB, which are transformed into daily data. Assuming the absence of arbitrage, we are able to estimate the parameters driving the dynamics of inflation under the objective probability measure and the corresponding objective probability distribution of inflation.

Our analysis is based on a factor model that – by exploiting the informative content of inflation swaps and inflation caps and floors – features a time-varying long-term mean and a stochastic volatility for inflation. These last two characteristics distinguish our model from those proposed in the literature for the euro area – see, among others, Camba-Méndez and Werner (2017), Casiraghi and Miccoli (2019) and Pericoli (2014). Our framework allows us to calculate the probability density function of inflation over a given horizon, not only under the risk-neutral probability measure, but also under the objective probability measure, net of the investors' risk aversion.[3] This offers the opportunity to calculate the objective probability that inflation is below or above a certain threshold. It is worth pointing out that the objective expected inflation implied in the model is anchored to analyst surveys in order to link the model results with observed data;[4] to the best of our knowl-

[2]See Abrahams et al. (2016) and references therein.

[3]Risk-neutral probabilities are probabilities of potential future outcomes adjusted for risk. The term risk-neutral can sometimes be misleading because some people may assume it means that the investors are neutral, unconcerned, or unaware of risk. In contrast, the risk-neutral probability accounts for the investors' aversion to risk: in general, risk-neutral probabilities tend to assign more weight to outcomes investors are worried about, attributing higher probability to extreme events such as deflation or very high inflation. Mathematically, the risk-neutral probability is the implied probability measure derived from the observable prices of the relevant instruments, defined using a risk-neutral utility function and assuming absence of arbitrage. On the other hand, objective probabilities are usually inferred from historical data, being estimated from the past dynamics of prices and other financial variables.

[4]We impose that the estimates are close to the inflation expected by analysts surveyed by the

edge, this is the first paper that incorporates these features, providing an innovative contribution to literature on euro-area inflation along several dimensions.

Our approach follows the traditional literature on factor models for estimating the term structure of interest rates and assumes that three factors are able to provide a good representation of inflation fluctuations. These factors are "instantaneous inflation",[5] "long-term inflation" and "expected volatility of inflation". But contrary to the traditional approach, the long-term average of inflation and the volatility of inflation can vary over time, as in Fleckenstein et al. (2017). Furthermore, by modeling inflation options, we can provide the entire probability distribution of inflation under the risk-neutral and objective measure and calculate the probability that inflation will be below or above a certain value over a certain horizon.

Our study is of particular interest for both policymakers and investors. The former can estimate the objective inflation expectations, evaluate the degree of anchoring of these expectations to the central bank's target and possibly assess the sensitivity of medium-to-long term inflation expectations to shocks affecting short-term ones and to inflation surprises (Miccoli and Neri, 2015; Corsello et al., 2021). The latter can use the measure of expected inflation to recover the discount rate in real terms to value fixed-income portfolios over long horizons.

This paper links to a large flow of literature on expected inflation, on the risk of deflation and the decoupling between expected short-term and long-term inflation. The closest paper to ours is Fleckenstein et al. (2017) in which the authors, based on the model by Heston (1993), extract the objective distribution of US inflation using prices of inflation derivatives and provide estimates of the objective probability of tail events and the inflation risk premium. Our main innovation compared to their work consists of anchoring the objective expected inflation implied in the model to analyst surveys. Cecchetti et al. (2015) use the information content of the inflation options to estimate risk neutral densities of inflation using the methodology introduced by Taboga (2016) and investigate signals of disanchoring of risk-neutral expectations. The authors go beyond the evidence that can be deduced simply by observing the moments of the distribution and propose to use different measures of the comovement between the moments of short-term and long-term risk-neutral distributions of inflation.[6]

The specification in this paper follows both the Heston (1993) stochastic volatility option pricing model and the long-run risk consumption model of Bansal and Yaron

central bank, for the available maturities.

[5] In a continuous time model, this is the annualized continuously compounded inflation rate that we expect to be realized for an infinitesimally short period of time, one day in our analysis.

[6] They find that, since mid-2014, negative tail events affecting short-term inflation expectations have increasingly been channelled towards long-term views, triggering both downward revisions in expectations and upward shifts in uncertainty; on the other hand, the short-term positive tail events mostly left the long-term moments unchanged. This asymmetrical impact may signal a bottom-up disanchoring in long-term inflation expectations.

(2004). Joining these models, we obtain as in Fleckenstein et al. (2017) a framework where inflation can have fluctuating uncertainty and a small, predictable long-term component. In general, this allows for a wide range of possible time-series properties for realized inflation and the inflation risk premium.

The literature on expected inflation and inflation risk premium is enormous. Typically, researchers have studied the breakeven inflation implied in nominal coupon bonds and in index-linked coupon bonds – see Abrahams et al. (2016), Adrian et al. (2013), Buraschi and Jiltsov (2005), Christensen et al. (2016), Christensen et al. (2012), Wright (2014) for a review – or one measured by inflation swaps – see Haubrich et al. (2012), Camba-Méndez and Werner (2017), Fleckenstein et al. (2017). As we mentioned above, we refer to this last field but we also link our objective measures of expected inflation to analysts' surveys in the spirit of Joyce et al. (2010) and Kim and Orphanides (2012). Regarding the estimate of the inflation risk premium, in the literature there are several models that have given different results in terms of magnitude and even sign – see Adrian et al. (2013), Pericoli (2014), Casiraghi and Miccoli (2019).

In a nutshell, our results show that the introduction of a time-varying long-term average inflation and variable volatility of inflation improves the current estimate of long-term expected inflation, that is close to 2%, the aim of the ECB in the medium term. Our estimates cover the period between October 2009 and December 2021. Long-term objective expected inflation was close to the ECB's target from 2010 to 2014 but subsequently declined, with temporary increases due to new waves of unconventional monetary policies, reaching a minimum in March 2020; after that this trend reversed and long-term objective expected inflation markedly increased in particular in the second half of 2021, climbing over 2% since September. The inflation risk premium, which was positive until 2014, became negative since 2015 and reached a minimum after the outbreak of the pandemic in 2020, to return to values close to zero in autumn 2021. The probability of inflation being negative over a 3-year horizon peaked above 50% in late 2014 and in early 2020, with the outbreak of the pandemic. Conversely, the likelihood that inflation could exceed the ECB's 2% target was always below 40%, except for the 2011-12 period, until September 2021, after which it rose to very high values, exceeding 80% in December.

The paper is structured as follows. Section (2) presents the model, the identification strategy and the data. Section (3) documents the results of our estimates while Section (4) compares them with those already in the literature. Section (5) concludes.

2 The model

We follow Bansal and Yaron (2004) and Fleckenstein et al. (2017) and estimate the following model

$$
\begin{aligned}
i(X,Y,T) &= A(T) + B(T)\cdot X + C(T)\cdot Y + \varepsilon(T)\ , && (1)\\
m(V,T) &= G(T) + H(T) + U(T)\cdot V + e(T)\ , && (2)
\end{aligned}
$$

where $i(X,Y,T)$ is the inflation swap rate with maturity T,[7] also referred to as i_T below, m is the volatility estimated from the risk-neutral density implied in options with expiration T. X, Y, V are three factors that drive inflation swaps and the option volatility. $A(T)$, $B(T)$, $C(T)$, $G(T)$, $H(T)$, $U(T)$ are vectors to be estimated. In keeping with the spirit of term structure models, these vectors have a recursive structure – see appendix. We consider 15 maturities for inflation swaps, i.e. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 25, 30, and 10 maturities for options volatility, i.e. 1, 2, 3, 5, 7, 10, 12, 15, 20, 30. The model we estimate, that is the formula for the swap price and the volatility of the options, is obviously determined by the dynamics of the factors and we report in the appendix both the derivation of the swap price and the implied volatility of the option, and in particular why we can separate $G(T)$ and $H(T) + U(T)\cdot V$ in equation (2). The dynamics of inflation dI, where I is the consumer price index, and the three factors under the objective probability measure $\mathbb{P}$ is

$$
\begin{aligned}
dI &= X\cdot Idt + \sqrt{V}I\cdot dZ_I^{\mathbb{P}} && (3)\\
dX &= \kappa(Y-X)dt + \eta dZ_X^{\mathbb{P}} && (4)\\
dY &= (\mu-\xi Y)dt + s dZ_Y^{\mathbb{P}} && (5)\\
dV &= (\delta-\psi V)dt + \sigma\sqrt{V}dZ_V^{\mathbb{P}} && (6)
\end{aligned}
$$

where $Z_I^{\mathbb{P}}, Z_X^{\mathbb{P}}, Z_Y^{\mathbb{P}}, Z_V^{\mathbb{P}}$ are uncorrelated Brownian motions. The model (3-6) incorporates the factor X, which represents istantaneous expected inflation, that tends to factor Y, which represents the long-run trend of inflation. Furthermore, inflation dI is determined non only by X and indirectly by Y but also by V, a variance factor that follows a stochastic process. The setting is reminiscent of the Bansal and Yaron (2004) model for istantaneous and long-run consumption and the Heston (1993) model for interest rates with stochastic volatility. In particular, the volatility of inflation has two components: the volatility due to the variation in expected inflation X_t and the volatility resulting from unexpected inflation, driven by the state variable V_t. The model (3-6) has a counterpart under the risk-neutral $\mathbb{Q}$ probability

[7]The inflation swap rate $i(X,Y,T)$ is equal to the expected inflation rate over the horizon T under the risk neutral measure $\mathbb{Q}$, i.e. $i(X,Y,T) = \mathbb{E}^Q[I_T/I_0 - 1]$ where I_k is the consumer price index at time k. It relates to the swap price $F(X,Y,T)$ with the formula $F_T = (1+i_T)^T$.

measure, i.e.:

$$\begin{aligned} dI &= X \cdot Idt + \sqrt{V} I dZ_I^{\mathbb{Q}} && (7)\\ dX &= \lambda(Y - X)dt + \eta dZ_X^{\mathbb{Q}} && (8)\\ dY &= (\alpha - \beta Y)dt + s dZ_Y^{\mathbb{Q}} && (9)\\ dV &= (\theta - \phi V)dt + \sigma\sqrt{V} dZ_V^{\mathbb{Q}} && (10) \end{aligned}$$

The connection between (7-10) and (3-6) is guaranteed by the assumption of a system of market prices of risk[8] that allows to obtain for each state variable the same dynamics under both probability measures, imposing standard conditions to exclude arbitrage opportunities. Note that even if the functional form of the drift for the I process is the same under the objective – equation (3) – and risk-neutral measures – equation (7) – this does not imply that the expected value is the same under $\mathbb{P}$ and $\mathbb{Q}$, because it is related to the different dynamics of X, Y and V.

The model (7-10) states that inflation can be written as

$$\begin{aligned} I_T/I_0 &= \exp(w_T + u_T) && (11)\\ w_T &= \int_0^T X_t dt \\ u_T &= -\frac{1}{2}\int_0^T V_t dt + \int_0^T \sqrt{V_t} dZ_I \end{aligned}$$

where the mean and the variance of w_T and u_T are shown in the appendix.

We use the Heston (1993) model to derive the density of u_T that we use to price options.

We also estimate the parameters of the processes (3-6) under the objective probability measure to retrieve expected inflation and deflation probabilities under this measure.

2.1 Identification

The factors are obtained using a completely standard approach, which is entirely based on the existing methodology widely applied in the literature. Following Chen and Scott (1993), Duffie and Singleton (1997) and Duffee (2002) we solve for the values of X, Y, and V from specific inflation swaps and option volatilities, and then jointly estimate the parameters of both the risk-neutral and objective dynamics for these variables using maximum likelihood. The unobservable factors are extracted by inverting the measurement equation by assuming that a number of assets equal to the number of factors is observed without error. In particular, we assume that

[8] See the Appendix for the derivation of the system of market prices of risk and the relationships between the drift parameters under the two probability measures.

the two inflation swaps with a maturity of 2 and 30 years and the implied volatility in the prices of the 3-year options are priced without errors,[9] namely $\varepsilon_2 = 0$, $\varepsilon_{30} = 0$, $e_3 = 0$. Therefore, the three factors, X, Y, V, are obtained by inverting the linear system (1-2) for $i = 2, 30$ and $j = 3$. The other inflation swaps and volatilities are priced with errors.

Furthermore, in order to estimate a reasonable value of expected inflation, we anchor the objective expected inflation to analysts surveys; specifically, we assume that the 1-year forward inflation rate in 0, 1 and 4 years time[10] implied in the inflation swaps estimated under the $\mathbb{P}$ measure, $f^{\mathbb{P}}(X, Y, l)$ with $l = 0, 1, 4$, are close, up to a measurement error, to the 1-year inflation expected by analysts surveyed by the central bank, $E^{SPF}(I_{l+1}/I_l)$, i.e.

$$f^{\mathbb{P}}(X, Y, l) = E^{SPF}(I_{l+1}/I_l) + z_l \ , \quad \text{for } l = 0, 1, 4 \ . \tag{12}$$

We assume that the errors have distribution $\varepsilon \sim N(0, \Sigma)$, $e \sim N(0, \Psi)$ and $z \sim N(0, \Omega)$ and that the covariance matrices are diagonal, i.e. the errors are uncorrelated, and denote the main diagonal elements with Σ_i, Ψ_j, Ω_l. Then, we maximize the log-likelihood function (13) conditional on the data over the 38-dimensional parameter vector $\Theta = \{\alpha,\ \beta,\ \lambda,\ \kappa,\ \mu,\ \xi,\ \sigma,\ s,\ \eta,\ \phi,\ \psi,\ \delta,\ \theta,\ \Sigma_1,\ \Sigma_3,\ \Sigma_4,\ \Sigma_5,\ \Sigma_6,\ \Sigma_7,\ \Sigma_8,\ \Sigma_9,\ \Sigma_{10},\ \Sigma_{12},\ \Sigma_{15},\ \Sigma_{20},\ \Sigma_{25},\ \Psi_1,\ \Psi_2,\ \Psi_5,\ \Psi_7,\ \Psi_{10},\ \Psi_{15},\ \Psi_{20},\ \Psi_{25},\ \Psi_{30},\ \Omega_1,\ \Omega_2,\ \Omega_5\}$ using a standard simplex algorithm.

$$\begin{aligned} L \ = \ & -(25/2)\ln(2\pi) + \ln(k) + \ln|J_{t+\Delta t}| - \frac{1}{2}\ln|\Sigma| - \frac{1}{2}\varepsilon'_{t+\Delta t}\Sigma^{-1}\varepsilon_{t+\Delta t} \\ & -\frac{1}{2}\ln|\Psi| - \frac{1}{2}e'_{t+\Delta t}\Psi^{-1}e_{t+\Delta t} - \frac{1}{2}\ln|\Omega| - \frac{1}{2}z'_{t+\Delta t}\Omega^{-1}z_{t+\Delta t} \\ & -\ln\left(2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}\right) - \frac{1}{2\left(1-\rho_{XY}^2\right)}\left[\left(\frac{X_{t+\Delta t}-\mu_{X_t}}{\sigma_X}\right)^2\right. \\ & \left. -2\rho_{XY}\left(\frac{X_{t+\Delta t}-\mu_{X_t}}{\sigma_X}\right)\left(\frac{Y_{t+\Delta t}-\mu_{Y_t}}{\sigma_Y}\right) + \left(\frac{Y_{t+\Delta t}-\mu_{Y_t}}{\sigma_Y}\right)^2\right] \\ & -k(V_{t+\Delta t} + V_t e^{-\psi\Delta t}) + \frac{1}{2}q(\ln V_{t+\Delta t} - \ln V_t + \psi\Delta t) \\ & +\ln \mathbf{I}_q\left(2k\sqrt{V_{t+\Delta t}V_t e^{-\psi\Delta t}}\right) \end{aligned} \tag{13}$$

where $\mathbf{I}_q(\cdot)$ is the modified Besseli function of the first kind of order $q = 2\delta/\sigma^2 - 1$ and $k = 2\psi/(\sigma^2(1 - e^{-\psi\Delta t}))$. J is the Jacobian of the linear mapping from the two inflation swaps and the implied volatility into X, Y and V. Note that, as usual in this literature, the errors terms ε, e and z in equation (13) are valued under the $\mathbb{Q}$ measure while the three factors, X, Y, V, under the $\mathbb{P}$ measure.

[9] For the choice of the assets assumed perfectly priced we follow Fleckenstein et al. (2017)

[10] These are the maturities for which survey data are available

This methodology makes it possible to jointly estimate the parameters of both the risk-neutral and objective dynamics and to recover the price of the risk inherent in the prices of inflation securities. We test for the existence of risk premia by examining whether the five parameters that appear in the objective dynamics of dI are equal to the corresponding parameters in the risk-neutral dynamics.

2.2 Option prices

We do not directly use caps and floors quotes for the estimation of model (1, 2, 12), while we use the standard deviation calculated from option price risk-neutral density function as in Cecchetti et al. (2015). Once the parameters have been estimated, we can retrieve the probability density function also under the objective measure $\mathbb{P}$, and consequently the corresponding objective prices.

Cap, C, and floor, P, prices are defined by

$$\begin{aligned} C(X,Y,V,T;K) &= D(T)\cdot E^{\mathbb{Q}^*}[\max(0,(1+i_T)^T-(1+K)^T)] \\ P(X,Y,V,T;K) &= D(T)\cdot E^{\mathbb{Q}^*}[\max(0,(1+K)^T-(1+i_T)^T)] \end{aligned}$$

where $D(T)$ is the discount factor and $\mathbb{Q}^*$ is a forward measure.[11] Under $\mathbb{Q}^*$, w_T is normally distributed with mean $\mu_u = \ln((1+i_T)^T) - \frac{1}{2}\sigma_w^2$ and variance σ_w^2, and u_T has a known distribution function $h(u_T)$ that we recover as a special case of the Heston (1993) model, with mean μ_u and variance σ_u^2. The discount factor $D(T)$ is defined as

$$D(T) = E^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}\right] \tag{14}$$

where r_t is the nominal instantaneous riskless interest rate, which is the sum of the real riskless interest rate R_t and instantaneous expected inflation X_t:

$$r_t = R_t + X_t. \tag{15}$$

Assuming that R_t and X_t are uncorrelated and that $R_t = 0$, cap and floor prices can be written as (see appendix)

$$C = (1+i_T)^{-T}\left\{\int_{-\infty}^{+\infty}\left[(1+i_T)^T N(a_1)e^{u_T} - (1+K)^T N(a_2)\right]h(u_T)du_T\right\} \tag{16}$$

$$P = (1+i_T)^{-T}\left\{\int_{-\infty}^{+\infty}\left[(1+K)^T N(-a_2) - (1+i_T)^T N(-a_1)e^{u_T}\right]h(u_T)du_T\right\} \tag{17}$$

where

$$a_1 = \frac{u_T - T\ln(1+K) + T\ln(1+i_T) + \frac{1}{2}\sigma_w^2}{\sqrt{\sigma_w^2}},$$

[11] The forward measure $\mathbb{Q}^*$ is defined by the dynamics of I given by equations (7), (8), (9) and (10), where the drift in (8) is augmented by $\eta^2 B(\tau)dt$ and the drift in (9) is augmented by $s^2 C(\tau)dt$, with $\tau = T - t$. See Brigo and Mercurio (2006) for a detailed explanation of the convenient definition of the forward definition.

$$a_2 = a_1 - \sqrt{\sigma_w^2} \ ,$$

K is the strike price and $N(\cdot)$ is the normal cumulative distribution function.

We confirm the correctness of the pricing formula for swaps, caps and floors in figures (1), (2) and (3) showing the convergence of a Monte Carlo simulation with 3,000 extractions. The results show that the simulation replicates after less than 1,000 steps the prices obtained by the closed formulas (1), (16) and (17)[12] .

We also verify that the prices of the options implied in the model are close to the observed prices of the options.

Furthermore, we use the Gram-Charlier expansion to approximate the distribution function of the logarithm of inflation $\ln I_T = w_T + u_T$. Define x the standardized value of inflation, with probability density function $f(x)$ and the first two cumulants $c_1 = \mu_w + \mu_u$ and $c_2 = \sigma_w^2 + \sigma_u^2$. The density function $f(x)$ can be approximated by

$$f(x) \approx \left[1 + c_1 x + \frac{1}{2}(c_1^2 + c_2 - 1)\left(x^2 - 1\right)\right] \cdot n\left(x\right) \tag{18}$$

where $n(\cdot)$ is the normal probability density function and $[1, x, x^2 - 1]$ are the Hermite polynomials up to the second order. c_1 and c_2 are functions of the parameters of model (1, 2, 12) and are shown in the appendix.

2.3 Data

We use daily data from October 2009 to December 2021 for inflation swaps with maturities 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 25, and 30 years.

Instead of using the full spectrum of option quotes with the corresponding maturity and strike prices, we summarize for each set of options with the same maturity their volatility by calculating the standard deviation of the inflation risk-neutral density function obtained from the option prices. This methodology makes it possible to reduce the number of parameters and to use a single time series of volatility per maturity, ignoring the presence of possible non-linearity in prices. We estimate the daily density function using zero caps and zero floors on euro-area HICP with maturities 1, 2, 3, 5, 7, 10, 12, 15, 20, 30 years and 17 strikes that range from -3% to 6%, for a total of around 170 time series. The density functions are estimated with the Cecchetti et al. (2015) and Taboga (2016) methodology and the standard deviation of the density is obtained by numerical integration.

The analysts' forecasts are derived from the quarterly survey of professional forecasters (SPF) conducted by the ECB, which are transformed into daily data assuming that the forecasts are constant until the new release. From SPF we use 1-year expected inflation, 1-year forward expected inflation after one year and the

[12]For the swap, the Monte Carlo exercise shows the convergence of the swap price $F_T = (1+i_T)^T$.

mean of the aggregate probability distribution of 1-year forward expected inflation after 4 years.

3 Results

3.1 Parameters and fitting

The estimates of the parameters of model (1, 2, 12) and the standard deviations – calculated with the Huber sandwich estimator – are presented in Table (1). The 38 parameters are highly significant with low p-values, except for the parameter θ, related to the long-term mean ($\frac{\theta}{\phi}$) of the volatility under the $\mathbb{P}$-measure. Notably, parameters under the $\mathbb{Q}$ measure and the $\mathbb{P}$ measure are significantly different. In particular, as regards to the factor V, the speed of mean reversion ψ is much larger under the $\mathbb{P}$ measure that the corresponding parameter ϕ under the $\mathbb{Q}$ measure, resulting in a volatility under the $\mathbb{P}$ measure lower than that under the $\mathbb{Q}$ measure. Morevoer, looking at the process of the factor Y corresponding to the long-run trend of inflation, the long-term mean under the $\mathbb{P}$ measure ($\frac{\mu}{\xi}$) is 1.98%, while the corresponding long-term mean under the $\mathbb{Q}$ measure ($\frac{\alpha}{\beta}$) is 2.98%.

Table (2) reports the pricing errors and the root mean squared errors for the inflation swaps and Table (3) the counterparties for the implied volatilities. Overall, the estimates of model (1, 2, 12) give a good approximation of the prices for both inflation swaps (with a pricing error in the range $[-5.15, +7.50]$ basis points) and implied volatility (with a pricing error in the range $[-5.42, +4.75]$ basis points). The goodness of fit can be valued by Figure 4 for inflation swaps and Figure 5 for implied volatility. The fitting for the former is particularly good, while for the latter we observe increasing errors as the maturity increases. This result is due to the fact that one factor may not be sufficient to cross-sectionally fit the term structure of implied volatilities.

In the spirit of the standard affine term structure models, we present the vectors A, B and C, which appear to have the usual form interpretable as level, slope and curvature and the vectors G, H and U, which are the corresponding loadings for the factor V (Figure 6).

3.2 Estimated factors and implied volatility

We report the estimates of the factors X and Y. The istantaneous expected inflation X shows large fluctuations from 2010 to today with a first peak in 2011 at 2.4%, a decline to -0.8% in 2015 and to -0.7% in 2020, and a maximum in December 2021 at 3.1% (Figure 7). Longer-term inflation Y, on the other hand, remained more stable around 2% from 2010 to 2015, falling to 0.8% in 2016, rising around 1.5% in the 2017-2019 period, decreasing to below 1% from mid 2019 to early 2021, and finally rising to 2% in autumn 2021.

We also report the estimates of the implied volatility m in equation (2) under the measure $\mathbb{Q}$ and $\mathbb{P}$ (Figure 8). For all maturities it is obtained that those under the $\mathbb{P}$ measure are lower than those under the $\mathbb{Q}$ measure, which means that objective inflation tends to be less volatile than breakeven inflation.

3.3 Expected inflation and inflation risk premium

We use the model (1, 2, 12) to calculate inflation swaps under the $\mathbb{P}$ measure – also labeled as expected inflation – and the inflation risk premium. The results for the 1-year, 3-year, 5-year, 10-year and 5-year five year forward maturity are presented in Figure 9. Let us take the 3 and 5-year maturity as representative of the medium-term outlook. For the 3-year maturity, the inflation risk premium, i.e. the difference between the market value of the inflation swap and the model value of the inflation swap estimated under the $\mathbb{P}$ measure, averages around 20 basis points from 2010 to 2015, drops to zero from 2015 to 2016, becomes negative between 2016 and 2017, goes back to zero until 2019, becomes negative again until the last quarter of 2021, and finally rises to approach zero. For the 5-year maturity, the dynamics of the inflation risk premium is similar but the size is larger in absolute values.

The results show that the introduction of a variable long-term inflation improves our understanding of expected inflation as we are able to achieve large variability in expected inflation and inflation risk premium at the same time.[13] Expected inflation over the long term was close to the ECB's target from 2010 to 2014 but has subsequently declined reaching a minimum in March 2020, with temporary increases as the ECB adopted quantitative measures to avoid the materialization of a deflationary scenario:[14] in particular, at the beginning of 2015 with the launch of the Asset Purchase Programme (APP), in the first quarter of 2016 with the increase in the pace of monthly purchases of government bonds under the APP from 60 to 80 billion euros, and early 2020 with the launch of the Pandemic Emergency Purchase Programme (PEPP). Thereafter, the declining trend reversed and long-term objective expected in‡ation markedly increased in particular in the second half of 2021, climbing over 2% since September.

At the same time, the inflation risk premium, positive until 2014, became negative since 2015 and reached a minimum after the outbreak of the pandemic in early 2020, to go back to values close to zero in autumn 2021 – see Bulligan et al. (2021) for a discussion of the comovement between the sign of the inflation risk premium and the correlation between inflation expectation and expected growth.

[13] This peculiarity is not obtained with other commonly used models; see section 4 for a comparison with other results available in the litarture.

[14] See **?** for estimates of survey- and market-based measures of inflation expectations and a discussion of their dynamics in recent years.

The inflation risk premium increases with maturity in absolute values but on average shows negative values across the maturity spectrum between 2015 and 2017 and since 2019 (with some exceptions in the last months of 2021). Figure 10 shows the average inflation risk premium and the 5th and 95th percentile in relation to maturity. Considering the 5-year maturity as representative, compared to an average value of approximately -2.5 basis points, the 5th percentile of the inflation risk premium is equal to -36 basis points and the 95th percentile to 26 basis points. Overall, the average value of the inflation risk premium between 2009 and 2021 is increasing in relation to maturities since the 10-year maturity but remains at modest levels, ranging between -4 and 12 basis points.

3.4 Comparison with survey forecasts

We compare expected inflation estimated under the $\mathbb{P}$ measure with that surveyed by the ECB SPF for the 1-year forward maturity in 1-year time and in 4-year time[15] (Figure 11). It is well known that in the last years the expected inflation measured by SPF, that is considered an objective measure of expected inflation (net of the risk premium component), has been higher than breakeven inflation (priced in traded assets and thus including inflation risk premium), and this has generated a negative inflation risk premium. Expected inflation detected by the SPF was also almost always higher than the expected one estimated by our model for the two maturities until the end of 2020, such that the inflation risk premium implied by the SPF surveys was not only negative but also lower than that estimated by the model; in the last year instead, the SPF estimates of expected inflation were lower than our objective estimates. Fitted annual inflation expected in 1 year closely follows break-even inflation producing a small and stable inflation risk premium, which however turns almost always negative from 2015. Furthermore, the estimated inflation expected in 1 year is close to SPF expected inflation up to half 2012, since then the former tends to decline more compared to the latter until the beginning of 2021, when the objective model estimates become higher than the survey counterparts. For the estimated inflation expected in 4 years, adjusted expected inflation and SPF expected inflation differ from 2012 but are generally closer to each other than the counterparts expected in 1 year. The difference between the two objective estimates of inflation expected in 4 years is negative between half 2012 and end 2020 and changes sign becoming positive later; in general, this difference widens when inflation decreases (as between end 2012 and 2016, and between end 2018 and the first quarter of 2020) while it decreases when inflation rises. At the end of 2021 however, after the marked rise of inflation, the positive difference between the model objective estimate (2.07%) and the survey estimate (1.85%) is around 22 basis points.

[15] They represent one-year inflation expected in one year and four years, respectively.

3.5 Probability density functions and probability of deflation/inflation

The model allows to calculate the probability density function under the measure $\mathbb{P}$ using the approximation presented in equation (18). Figure 12 shows the densities for the 2, 3, 5, and 10 year maturities. These densities allow us to calculate the probability that inflation falls within selected ranges over a defined horizon under measure $\mathbb{P}$. Since the risk of observing deflation or exceeding the ECB target are among the most debated topics within the economic analysis debate, we report, on the one hand, the probability that inflation is negative and, on the other hand, the probability that it exceeds 2% over a three-year horizon under both measures $\mathbb{P}$ and $\mathbb{Q}$ in Figure 13.

The probability under measure $\mathbb{P}$ that inflation is negative in three years is negligible from 2009 to mid-2014 and from 2017 to early 2020. It exceeds 50% in late 2014 and in early 2020 with the outbreak of the pandemic. Since 2009, the probability under measure $\mathbb{P}$ is lower than that under measure $\mathbb{Q}$ during quiet periods, while it is higher during times of crisis. The objective probability of inflation exceeding the ECB target of 2% is less than 40%, except for the 2011-12 period when it reaches 70%, until September 2021; in the last quarter of 2021 it rose to very high values, exceeding 80 per cent in December. In general, the probability of inflation exceeding 2% is similar in the two measures but looking at the peaks observed at the beginning and at the end of the review period, while in the first the probability under measure $\mathbb{P}$ was lower than that under measure $\mathbb{Q}$, in the most recent peak the probability under measure $\mathbb{P}$ was haigher than that under measure $\mathbb{Q}$.

4 Extension

4.1 Comparison with other models

We compare our results with those obtained from the models prevalent in the literature and among central banks. Figure 14 presents the expected inflation and the inflation risk premium for the 5-year maturity in five years estimated with the methodology of this paper (CGP), with those obtained by replicating the methodology of Adrian et al. (2013) (ACM), Joslin et al. (2011) (JSZ) and Pericoli (2014) (PER). The long-term average expected inflation estimated with the ACM and JSZ models is set at 1.9%, close to the ECB's long-term inflation target. The long-term average expected inflation estimated with the CGP and PER models is not constrained, as the long-term expected inflation is anchored to the analysts' inflation survey. CGP expected inflation is similar to that estimated with the PER methodology and extremely different from ACM Expected Inflation, which is surprisingly stable. JSZ expected inflation, a measure adopted by the ECB staff as published in Camba-Méndez and Werner (2017), is less variable and very close to CGP and

PER estimates. As for the inflation risk premium, the CGP premium has a similar dynamics to the ACM one, while is different from the PER premium, which is smoother and never drops to negative values. The JSZ risk premium is very similar to that estimated by CGP. Overall, the CGP estimates seem to provide a reasonable description of expected inflation thanks to the introduction of long-term variable average inflation avoid setting long-term expected inflation at a particular value.

5 Conclusion

We propose a factor model to measure expected inflation and the inflation risk premium that leverages two sets of market instruments, inflation swaps and inflation caps and floors, and that takes into account analysts' survey forecasts of inflation to anchor objective measures of inflation. The model specification, featuring variable long-term average inflation and variable inflation volatility, allows for a fairly general structure of inflation risk premia.

We use a specification developed in the seminal contributions of Bansal and Yaron (2004) and Fleckenstein et al. (2017) that nests the contribution of the Heston (1993) option pricing model with stochastic volatility. Our framework makes it possible to calculate the density function of inflation over a given horizon not only under the risk-neutral probability measure, but also under the objective probability measure. Furthermore it offers the opportunity to calculate the objective probability that inflation is below or above a certain threshold.

The results show that the introduction of time-varying average inflation and time-varying volatility of inflation improves the current estimate of long-term expected inflation, that is just below the inflation aim of the ECB in the medium term. In particular, long-term expected inflation was close to the ECB's target from 2010 to 2014 but has subsequently declined reaching a minimum in March 2020, with temporary increases due to new waves of unconventional monetary policies; thereafter, the declining trend reversed and long-term objective expected inflation markedly increased, in particular in the second half of 2021, climbing over 2% since September.As a result, the inflation risk premium, which was positive until 2014, became negative since 2015 and reached a minimum after the outbreak of the pandemic in early 2020, to go back to values close to zero in autumn 2021.

The probability of inflation being negative over a 3-year horizon peaked above 50% in late 2014 and early 2020 with the outbreak of the pandemic. Conversely, the likelihood that inflation could exceed the ECB's 2% target was always below 40%, except for the 2011-12 period, until September 2021; in the last quarter of 2021 it rose to very high values, exceeding 80% in December.

The model makes it possible to analyze the differences between the spot and forward inflation risk premium over different maturities and to investigate its contribution to changes in breakeven inflation, an issue at the center of the economic policy debate in the second half of 2021. To this end, the model allows us to investigate two questions raised in recent months not only by the ECB but also by the monetary authorities of the main advanced countries. First, it allows us to establish whether the increase in inflation expectations in the medium and long term is due to the positive effect of the re-anchoring of inflation expectations to the ECB target or to a worrying dis-anchoring of expectations. Second, it makes it possible to investigate which financial variables, macroeconomic variables or commodity prices impact the inflation risk premia and inflation expectations. We leave these projects to future research.

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A Appendix

A.1 Derivation of the system for the market prices of risk

Given a generic diffusion process $dW_t = h(W_t)dt + f(W_t)dZ^{\mathbb{P}}_{W,t}$, where $Z^{\mathbb{P}}_{W,t}$ is a standard Brownian motion under the objective $\mathbb{P}$-measure, the essentially affine market price of risk[16]

$$\Lambda^W_t = \gamma^W_0 + \gamma^W_1 g(W_t)$$

defines the relationship between $Z^{\mathbb{P}}_{W,t}$and the corresponding Brownian motion process under the risk-neutral $\mathbb{Q}$-measure $Z^{\mathbb{Q}}_{W,t}$, i.e.

$$Z^{\mathbb{Q}}_{W,t} = Z^{\mathbb{P}}_{W,t} + \int_0^t \Lambda^W_s ds = Z^{\mathbb{P}}_{W,t} + \int_0^t \gamma^W_0 + \gamma^W_1 g(W_s)ds$$

Note that, in case of a CIR process, we have $h(W_t) = k_0 + k_1 W_t$, $f(W_t) = \sigma\sqrt{W_t}$ and $g(W_t) = \sqrt{W_t}$. Assuming standard conditions to exclude arbitrage opportunities,[17] the relationship between Brownian motions allows to obtain under the $\mathbb{Q}$-measure the same dynamics followed by the state variable W under the $\mathbb{P}$-measure, by appropriately adjusting the drift parameters.[18] Since model (7-10) has three state

[16] See Cheridito et al. (2007) for a review of different specifications of market prices of risk.

[17] See Cheridito et al. (2007) for details.

[18] See the Appendix in Cecchetti (2020) for a simple derivation of the link between the risk neutral and objective dynamics for some stochastic processes, given the assumption of a proper market price of risk.

variables, i.e. $W = (X, Y, V)$, with different stochastic processes where X links to Y, the market price of risk is defined by a system that links the dynamics under the two probability measures.

The dynamics under the risk-neutral $\mathbb{Q}$-measure

$$
\begin{aligned}
d\begin{pmatrix} X \\ Y \\ V \end{pmatrix} &= \begin{pmatrix} 0 \\ \alpha \\ \theta \end{pmatrix} dt + \begin{pmatrix} -\lambda & \lambda & 0 \\ 0 & -\beta & 0 \\ 0 & 0 & -\phi \end{pmatrix} \begin{pmatrix} X \\ Y \\ V \end{pmatrix} dt + \\
& \begin{pmatrix} \eta & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \sigma\sqrt{V} \end{pmatrix} \begin{pmatrix} dZ_X^{\mathbb{Q}} \\ dZ_Y^{\mathbb{Q}} \\ dZ_V^{\mathbb{Q}} \end{pmatrix}
\end{aligned}
$$

can be written in terms of the dynamics under the objective $\mathbb{P}$-measure

$$
\begin{aligned}
d\begin{pmatrix} X \\ Y \\ V \end{pmatrix} &= \begin{pmatrix} 0 \\ \mu \\ \delta \end{pmatrix} dt + \begin{pmatrix} -\kappa & \kappa & 0 \\ 0 & -\xi & 0 \\ 0 & 0 & -\psi \end{pmatrix} \begin{pmatrix} X \\ Y \\ V \end{pmatrix} dt + \\
& \begin{pmatrix} \eta & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \sigma\sqrt{V} \end{pmatrix} \begin{pmatrix} dZ_X^{\mathbb{P}} \\ dZ_Y^{\mathbb{P}} \\ dZ_V^{\mathbb{P}} \end{pmatrix}
\end{aligned}
$$

as

$$
\begin{aligned}
d\begin{pmatrix} X \\ Y \\ V \end{pmatrix} &= \begin{pmatrix} 0 \\ \alpha \\ \theta \end{pmatrix} dt + \begin{pmatrix} -\lambda & \lambda & 0 \\ 0 & -\beta & 0 \\ 0 & 0 & -\phi \end{pmatrix} \begin{pmatrix} X \\ Y \\ V \end{pmatrix} dt + \\
& \begin{pmatrix} \eta & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \sigma\sqrt{V} \end{pmatrix} \left[\begin{pmatrix} dZ_X^{\mathbb{P}} \\ dZ_Y^{\mathbb{P}} \\ dZ_V^{\mathbb{P}} \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma_0^Y \\ \gamma_0^V/\sqrt{V} \end{pmatrix} dt + \right. \\
& \left. \begin{pmatrix} \gamma_1^X & -\gamma_1^X & 0 \\ 0 & \gamma_1^Y & 0 \\ 0 & 0 & \gamma_1^V/\sqrt{V} \end{pmatrix} \begin{pmatrix} X \\ Y \\ V \end{pmatrix} dt \right] \\
&= \begin{pmatrix} 0 \\ \alpha + s\gamma_0^Y \\ \theta + \sigma\gamma_0^V \end{pmatrix} dt + \\
& \begin{pmatrix} -\lambda + \eta\gamma_1^X & \lambda - \eta\gamma_1^X & 0 \\ 0 & -\beta + s\gamma_1^Y & 0 \\ 0 & 0 & -\phi + \sigma\gamma_1^V \end{pmatrix} \begin{pmatrix} X \\ Y \\ V \end{pmatrix} dt + \\
& \begin{pmatrix} \eta & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & \sigma\sqrt{V} \end{pmatrix} \begin{pmatrix} dZ_X^{\mathbb{P}} \\ dZ_Y^{\mathbb{P}} \\ dZ_V^{\mathbb{P}} \end{pmatrix}
\end{aligned}
$$

and the mapping for the drift parameters between the two measures is

$$\begin{aligned}\kappa &= \lambda - \eta\gamma_1^X \\ \mu &= \alpha + s\gamma_0^Y \\ \xi &= \beta - s\gamma_1^Y \\ \delta &= \theta + \sigma\gamma_0^V \\ \psi &= \phi - \sigma\gamma_1^V\end{aligned}$$

Note that the Feller condition for the CIR process governing the dynamics of V must apply for the solution to be bounded below by zero. Note also that $(\gamma_0^X, \gamma_0^Y, \gamma_0^V)$ and $(\gamma_1^X, \gamma_1^Y, \gamma_1^V)$ define the differences between the drift terms for the processes X, Y and V within the objective $\mathbb{P}$-measure and risk-neutral $\mathbb{Q}$-measure and allow the market to incorporate time-varying inflation risk premia into prices.

A.2 Inflation swap pricing

From equations (7) and (11) , the price index at time T can be written

$$I_T/I_0 = \exp\left(\int_0^T X_s ds - \frac{1}{2}\int_0^T V_s ds + \int_0^T \sqrt{V_s} dZ_{I,s}\right)$$

where we can set $I_0 = 1$ without loss of generality. The cash flow of an inflation swap is equal to $I_T - (1+i_T)^T$ and since the present value of the inflation swap is nil at inception, we can write

$$E^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right)\left(I_T - (1+i_T)^T\right)\right] = 0$$

We define the instantaneous nominal rate equal to the sum of the instantaneous real rate and expected inflation, $r_t = R_t + X_t$, and substituting r_t and I_T obtain

$$E^{\mathbb{Q}}\left[e^{-\int_0^T R_s ds}\right]\left\{E^{\mathbb{Q}}\left[e^{-\int_0^T X_s ds} e^{\int_0^T X_s ds - \frac{1}{2}\int_0^T V_s ds + \int_0^T \sqrt{V_s} dZ_{I,s}}\right] - E^{\mathbb{Q}}\left[e^{-\int_0^T X_s ds}(1+i_T)^T\right]\right\} = 0$$

which implies

$$(1+i)^T = \frac{E^{\mathbb{Q}}\left[\exp\left(-\frac{1}{2}\int_0^T V_s ds + \int_0^T \sqrt{V_s} dZ_{I,s}\right)\right]}{E^{\mathbb{Q}}\left[\exp\left(-\int_0^T X_s ds\right)\right]} = \frac{1}{E^{\mathbb{Q}}\left[\exp\left(-\int_0^T X_s ds\right)\right]} \tag{A.1}$$

If we set $E^{\mathbb{Q}}\left[\exp\left(-\int_0^T X_s ds\right)\right] = H(X, Y, \tau)$ where $\tau = T - t$, H satisfies the following PDE

$$\frac{1}{2}\eta^2 H_{XX} + \frac{1}{2}s^2 H_{YY} + \lambda(Y - X)H_X + (\alpha - \beta Y)H_Y - XH - \frac{\partial H}{\partial \tau} = 0$$

We guess a solution of the form $H = \exp(A(\tau) + B(\tau)X + C(\tau)Y)$ such that $H_{XX} = B^2H$, $H_{YY} = C^2H$, $H_X = BH$, $H_Y = CH$ and obtain

$$\begin{aligned}
\partial B(\tau)/\partial\tau &= -\lambda B(\tau) - 1 \\
\partial C(\tau)/\partial\tau &= \lambda B(\tau) - \beta C(\tau) \\
\partial A(\tau)/\partial\tau &= \frac{1}{2}\eta^2 B^2(\tau) + \frac{1}{2}s^2 C^2(\tau) + \alpha C(\tau)
\end{aligned}$$

The equation are solved by the use of an integrating factor and direct integration. We substitute the solutions into the expression for $H(X, Y, T)$ into equation (A.1) and evaluate at $\tau = T$. This gives equation (1).

A.3 Term structure of inflation swaps

The vectors $A(T), B(T), C(T)$ have the following expression under the $\mathbb{Q}$-measure:

$$A(T) = -\frac{1}{T}\cdot\left[\begin{array}{c}
\frac{\alpha\lambda}{\beta-\lambda}\left(\frac{1}{\beta}\left(T-\frac{1}{\beta}(1-e^{-\beta T})\right)-\frac{1}{\lambda}\left(T-\frac{1}{\lambda}(1-e^{-\lambda T})\right)\right) \\
+\frac{s^2\lambda^2}{2(\lambda-\beta)^2}\left(\frac{1}{\beta^2}\left(T-\frac{2}{\beta}(1-e^{-\beta T})\right)-\frac{1}{2\beta}\left(T-\frac{1}{\lambda}(1-e^{-2\beta T})\right)\right) \\
-\frac{2}{\beta\lambda}\left(T-\frac{1}{\beta}(1-e^{-\beta T})-\frac{1}{\lambda}(1-e^{-\lambda T})+\frac{1}{\beta+\lambda}(1-e^{-(\beta+\lambda)T})\right) \\
+\frac{1}{\lambda^2}\left(T-\frac{2}{\lambda}(1-e^{-\lambda T})+\frac{1}{2\lambda}(1-e^{-2\lambda T})\right) \\
+\frac{\eta^2}{2\lambda^2}\left(T-\frac{2}{\lambda}(1-e^{-\lambda T})+\frac{1}{2\lambda}(1-e^{-2\lambda T})\right)
\end{array}\right] \tag{A.2}$$

$$B(T) = -\frac{1}{T}\cdot\frac{-(1-e^{-\lambda T})}{\lambda} \tag{A.3}$$

$$C(T) = -\frac{1}{T}\cdot\frac{\lambda}{\beta-\lambda}\left(\frac{1}{\beta}(1-e^{-\beta T})-\frac{1}{\lambda}(1-e^{-\lambda T})\right) \tag{A.4}$$

The last four lines of equation (A.2) can be substituted by $\sigma_w^2(T)$ defined below by equation (A.6).

A.4 Distribution of inflation

Under the $\mathbb{Q}^*$-measure the two members of equation (11), w_T and u_T, have the following distribution. $w_T \sim N(\mu_w(T), \sigma_w^2(T))$ where

$$\mu_w(T) = (1 + i_T)^T - \frac{1}{2}\sigma_w^2(T) \tag{A.5}$$

and variance

$$
\begin{aligned}
\sigma_w^2(T) \quad = \quad & \frac{s^2\lambda^2}{(\lambda-\beta)^2}\left(\frac{1}{\beta^2}\left(T-\frac{2}{\beta}(1-e^{-\beta T})+\frac{1}{2\beta}(1-e^{-2\beta T})\right)\right. \qquad \text{(A.6)}\\
& -\frac{2}{\beta\lambda}\left(T-\frac{1}{\beta}(1-e^{-\beta T})+\frac{1}{\lambda}(1-e^{-\lambda T})+\frac{1}{\beta+\lambda}(1-e^{-(\beta+\lambda)T})\right)\\
& \left.\frac{1}{\lambda^2}\left(T-\frac{2}{\lambda}(1-e^{-\lambda T})+\frac{1}{2\lambda}(1-e^{-2\lambda T})\right)\right)\\
& +\frac{\eta^2}{\lambda^2}\left(T-\frac{2}{\lambda}(1-e^{-\lambda T})+\frac{1}{2\lambda}(1-e^{-2\lambda T})\right)
\end{aligned}
$$

The distribution of u_T is obtained from

$$
\begin{aligned}
du &= -\frac{1}{2}V dt + \sqrt{V} dZ_I\\
dV &= (\theta-\phi V)dt + \sigma\sqrt{V} dZ_V
\end{aligned}
$$

that is a special case of the Heston (1993) model with characteristic function $E[e^{i\zeta u_T}]$ of u_T given by

$$
\exp(L(T)+M(T)V) \qquad \text{(A.7)}
$$

where

$$
\begin{aligned}
L(T) &= \frac{\theta(\phi+\gamma)}{\sigma^2}T+\frac{2\theta}{\sigma^2}\ln\left(\frac{1-k_0}{1-k_0e^{\gamma T}}\right)\\
M(T) &= \frac{\phi+\gamma}{\sigma^2}\cdot\frac{1-e^{\gamma T}}{1-k_0e^{\gamma T}}
\end{aligned}
$$

and where

$$
\begin{aligned}
\gamma &= \sqrt{\sigma^2(\zeta^2+i\zeta)+\phi^2}\\
k_0 &= (\phi+\gamma)/(\phi-\gamma)
\end{aligned}
$$

Then, the density function of u_T, obtained by inverting the characteristic function, is given by

$$
h(u_T)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\zeta u_T}\exp(L(T)+M(T)V)d\zeta \qquad \text{(A.8)}
$$

The cumulants of u_T are obtained in closed form by repeatedly differentiating the log of the characteristic function (A.7) with respect to the argument ζ and evaluating the derivatives at $\zeta=0$. The first cumulant of u_T, i.e. the mean of u_T, is given by

$$
\mu_u(T)=-\frac{1}{2}\left(V-\frac{\theta}{\phi}\right)\frac{1}{\phi}(1-e^{-\phi T})-\frac{1}{2}\frac{\theta}{\phi}T \qquad \text{(A.9)}
$$

The second cumulant of u_T, the variance of u_T, is given by

$$\begin{aligned}
\sigma_u^2(T) &= \left(1+\frac{\sigma^2}{4\phi^2}\right)\left(-\frac{\theta}{\phi^2}(1-e^{-\phi T})+\frac{\theta}{\phi}T\right) \\
&\quad -\frac{\sigma^2}{2\phi^2}\left(-\frac{\theta}{\phi}e^{-\phi T}+\frac{\theta}{\phi^2}(1-e^{-\phi T})\right) \\
&\quad +\frac{\sigma^2}{4\phi^2}\left(-\frac{\theta}{\phi^2}(e^{-\phi}-e^{-2\phi T})+\frac{\theta}{2\phi^2}(1-e^{-2\phi T})\right) \\
&\quad +\left(\left(1+\frac{\sigma^2}{4\phi^2}\right)\frac{1}{\phi}(1-e^{-\phi T})-\frac{\sigma^2}{2\phi^2}e^{-\phi T}+\frac{\sigma^2}{4\phi^2}\frac{1}{\phi}(e^{-\phi T}-e^{-2\phi T})\right)V \quad .
\end{aligned} \tag{A.10}$$

The distribution of inflation is defined in terms of the joint density of u_T and that of w_T, that given their independence is the product of their marginals. The cumulants of the distribution of the logarithm of inflation are equal to the sum of the cumulants of w_T and that of u_T. We use the Gram-Charlier expansion to numerically approximate this density under the $\mathbb{P}$-measure given that we know the functional form and the moments of both components.

A.5 Gram-Charlier expansion

Denote the first and second cumulant of inflation as $c_1 = \mu_w(T) + \mu_u(T)$ – defined in equations (A.5,A.9) – and $c_2 = \sigma_w^2(T) + \sigma_u^2(T)$ – defined in equations (A.6,A.10) – see Chateau and Dufresne (2017) for references. Standardize inflation and define it x. The first three Hermite polynomials He_n, for $n = 1, 2, 3$, are given by

$$\begin{aligned}
He_0 &= 1 \\
He_1 &= x \\
He_2 &= x^2 - 1
\end{aligned}$$

By the definition of Gram-Charlier expansion, the density of x can be approximated up to the second order by

$$\begin{aligned}
f(x,T) &\approx \left[He_0 + c_1 He_1 + \frac{1}{2}\left({c_1}^2 + c_2 - 1\right) He_2\right]\cdot n(x) \\
&= \left[1 + c_1 x + \frac{1}{2}\left({c_1}^2 + c_2 - 1\right)(x^2-1)\right]\cdot n(x)
\end{aligned}$$

where $n(\cdot)$ is the density function of the standard normal. In order to overcome the usual problem of having negative values of the term in square brackets we use the following normalization

$$f(x,T) \cong \frac{\left[1 + c_1 x + \frac{1}{2}\left({c_1}^2 + c_2 - 1\right)(x^2-1)\right]^2}{\left[1 + {c_1}^2 + \frac{1}{2}({c_1}^2 + c_2 - 1)^2\right]}\cdot n(x)$$

The probabilities of observing an inflation below zero and above 2% are equal to

$$\Pr(x < 0) = \int_{-\infty}^{0} f(x,T)dx \tag{A.11}$$

$$\Pr(x > 0.02) = \int_{0.02}^{\infty} f(x,T)dx \ . \tag{A.12}$$

Figure 12 reports $f(x,T)$ for $T = 2, 3, 5, 10$. Figure 13 reports the probabilities (A.11-A.12) for $T = 3$.

A.6 Pricing of caps and floors

The no-arbitrage price of a cap option is given by the expectation under the risk-neutral measure $\mathbb{Q}$ of the discounted payoff:

$$\mathbb{E}^{\mathbb{Q}}\left[D(T) \cdot \max(0, I_T - (1+K)^T)\right] \ .$$

Applying the change-of-numeraire technique, this expectation can be computed as the product of the convenient numeraire, $D(T)$, and the expectation under the $\mathbb{Q}^*$ forward measure:

$$D(T) \cdot \mathbb{E}^{\mathbb{Q}^*}\left[\max(0, I_T - (1+K)^T)\right]$$

where

$$D(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T X_s ds}\right] = (1+i_T)^{-T}$$

and the expectation can be written as

$$\mathbb{E}^{\mathbb{Q}^*}\left[\max(0, (1+i_T)^T - (1+K)^T)\right] = \mathbb{E}^{\mathbb{Q}^*}\left[\max(0, e^{u_T + w_T} - (1+K)^T)\right]$$

as function of u_T and w_T whose joint density is $f(w_T, u_T) = f(w_T) \cdot f(u_T)$ by the independence between w_T and u_T. By direct integration:

$$\mathbb{E}^{\mathbb{Q}^*}\left[\max(0, e^{u_T + w_T} - (1+K)^T)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \max\left(0, e^{u_T + w_T} - (1+K)^T\right) f_{w_T} \cdot f_{u_T} dw_T du_T \ .$$

As $e^{u_T + w_T} > (1+K)^T$ if and only if $w_T > T\ln(1+K) - u_T$ the expectation can be written as

$$\int_{-\infty}^{+\infty}\int_{T\ln(1+K)-u_T}^{+\infty} e^{u_T} e^{w_T} f_{w_T} f_{u_T} dw_T du_T - (1+K)^T \int_{-\infty}^{+\infty}\int_{T\ln(1+K)-u_T}^{+\infty} f_{w_T} f_{u_T} dw_T du_T \ . \tag{A.13}$$

- Under the $\mathbb{Q}^*$ measure $w_T \sim N(\ln((1+i_T)^T) - \sigma_w^2/2, \sigma_w^2)$ with density,

$$f_W = \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{1}{2}\frac{(w_T - \ln((1+i_T)^T) + \sigma_w^2/2)^2}{G}}$$

and distribution function

$$F_W(w) = N\left(\frac{w - \ln((1+i_T)^T) + \sigma_w^2/2}{\sqrt{G}}\right)$$

- The first term of of (A.13) can be written as

$$\int_{-\infty}^{+\infty} e^{u_T} f_{u_T} \int_{\ln\frac{(1+K)^T}{e^{u_T}}}^{+\infty} e^{w_T} f_{w_T} dw_T du_T \tag{A.14}$$

Because $\int_{-\infty}^{+\infty} e^{u_T} f_{u_T} du_T = \mathbb{E}^{\mathbb{Q}^*}\left[e^{u_T}\right] = 1$, this term converges.
The term in w can be written as

$$\int_{\ln\frac{(1+K)^T}{e^{u_T}}}^{+\infty} e^{w_T} f_{w_T} dw_T = \mathbb{E}^{\mathbb{Q}^*}\left[e^{w_T}\mathbf{1}_{e^{w_T} > \frac{(1+K)^T}{e^{u_T}}}\right] \tag{A.15}$$

Since w_T is normal, e^{w_T} is lognormal with mean

$$\mathbb{E}[e^{w_T}] = e^{\ln((1+i_T)^T) - \sigma_w^2/2 + \sigma_w^2/2}$$

variance

$$Var[e^{w_T}] = (e^{\sigma_w^2} - 1)e^{2(\ln((1+i_T)^T) - \sigma_w^2/2) + \sigma_w^2}$$

and density

$$\frac{1}{\sqrt{\sigma_w^2 2\pi e^w}} e^{-\frac{1}{2}\left(\frac{(\ln e^w - \ln((1+i_T)^T) + \sigma_w^2/2)^2}{\sigma_w^2}\right)}$$

equation (A.15) becomes

$$\mathbb{E}^{\mathbb{Q}^*}\left[e^{w_T}\mathbf{1}_{e^{w_T} > \frac{(1+K)^T}{e^{u_T}}}\right] = \int_{\frac{(1+k)^T}{e^{u_T}}}^{\infty} \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{1}{2}\frac{(\ln(e^{w_T} - \ln((1+i_T)^T) + \sigma_w^2/2)^2}{G}} d(e^{w_T})\ .$$

By a change of variable $w = \ln(e^w)$, $d(e^w) = e^w dw$

$$\int_{T\ln(1+k) - u_T}^{\infty} e^{w_T} \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{1}{2}\frac{(w_T - \ln((1+i_T)^T) + \sigma_w^2/2)^2}{G}} dw_T\ . \tag{A.16}$$

Combining terms and completing the square, the exponent in (A.16) becomes:

$$\begin{aligned} & -\frac{1}{2\sigma_w^2}(w^2 + \ln((1+i_T)^T)^2 + \frac{\sigma_w^4}{4} - 2w\ln((1+i_T)^T) + w\sigma_w^2 - \ln((1+i_T)^T)\sigma_w^2 - 2\sigma_w^2 w) \\ = & -\frac{1}{2\sigma_w^2}\left(w - (\ln((1+i_T)^T) + \sigma_w^2/2)\right)^2 + \ln((1+i_T)^T) \end{aligned}$$

and defining $a = \frac{u_T - T\ln(1+K) + \ln((1+i_T)^T) + \sigma_w^2/2}{\sqrt{\sigma_w^2}}$, (A.16) can be written as

$$(1+i_T)^T \int_{T\ln(1+k) - u_T}^{\infty} \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{1}{2}\frac{(w_T - (\ln((1+i_T)^T) + \sigma_w^2/2))^2}{G}} dw_T = (1+i_T)^T \cdot N(a)\ \ .$$

- The second term of (A.13) can be written as

$$
\begin{aligned}
&-(1+K)^T \int_{-\infty}^{+\infty} f_{u_T}[1 - F_W(T\ln(1+K) - u_T)]du_T \\
&-(1+K)^T \int_{-\infty}^{+\infty} N(a - \sqrt{\sigma_w^2}) f_{u_T} du_T \quad .
\end{aligned}
$$

  where $[1 - F_W(T\ln(1+K) - u_T)] = N(a - \sqrt{\sigma_w^2})$.

- Combining the terms, the price of the cap (A.13) is given by

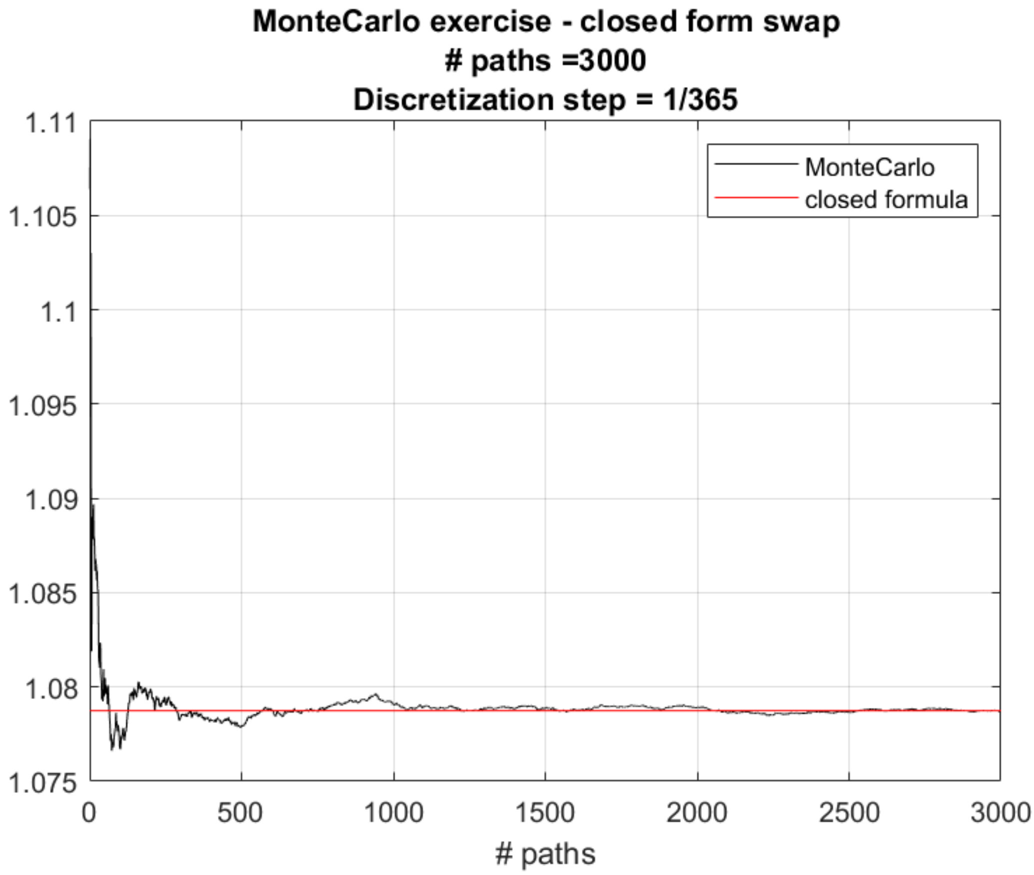
$$
(1+i_T)^{-T} \int_{-\infty}^{+\infty} ((1+i_T)^T \cdot N(a) \cdot e^{u_T} - (1+K)^T \cdot N(a - \sqrt{\sigma_w^2})) \cdot f_{u_T} du_T
$$

  The price of the floor option can be derived in a specular way.

For a different derivation, see also appendix B in Fleckenstein et al. (2017).

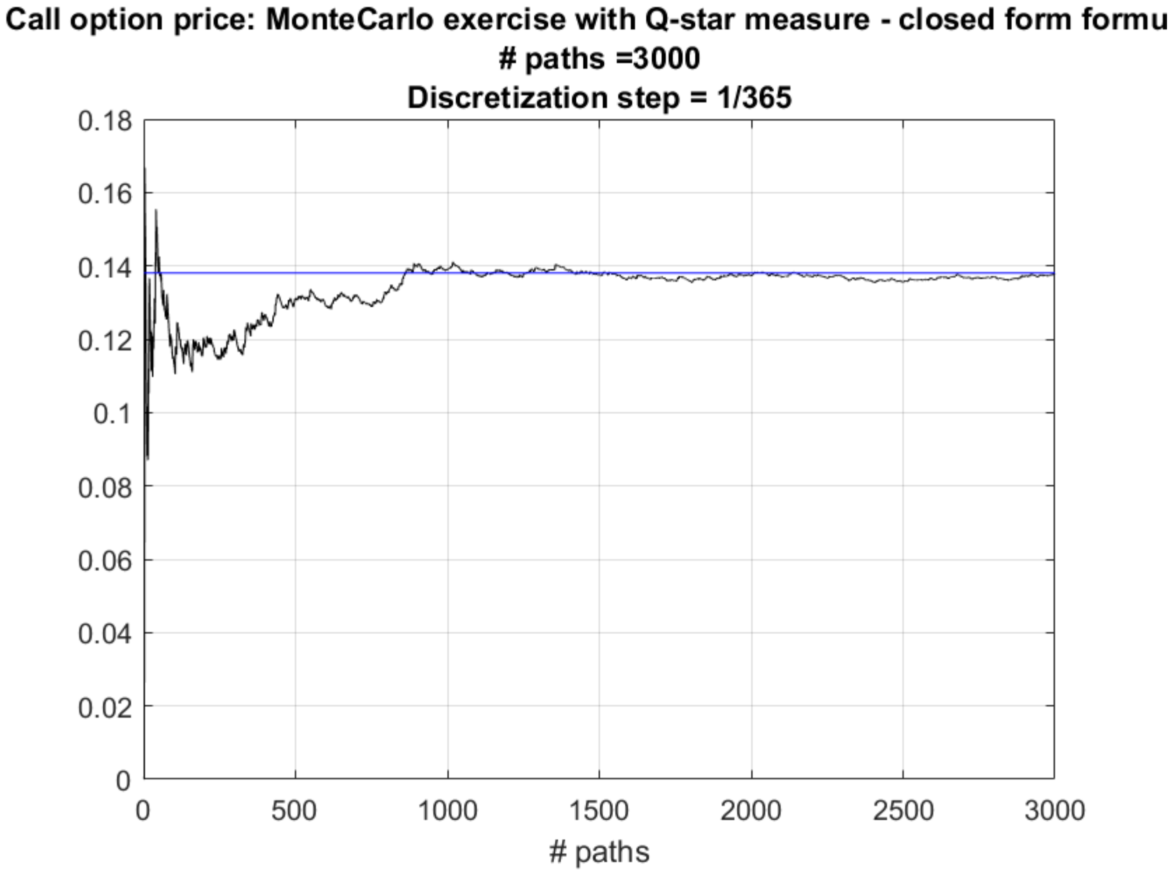
B Tables and figures

Figure 1: Swap price



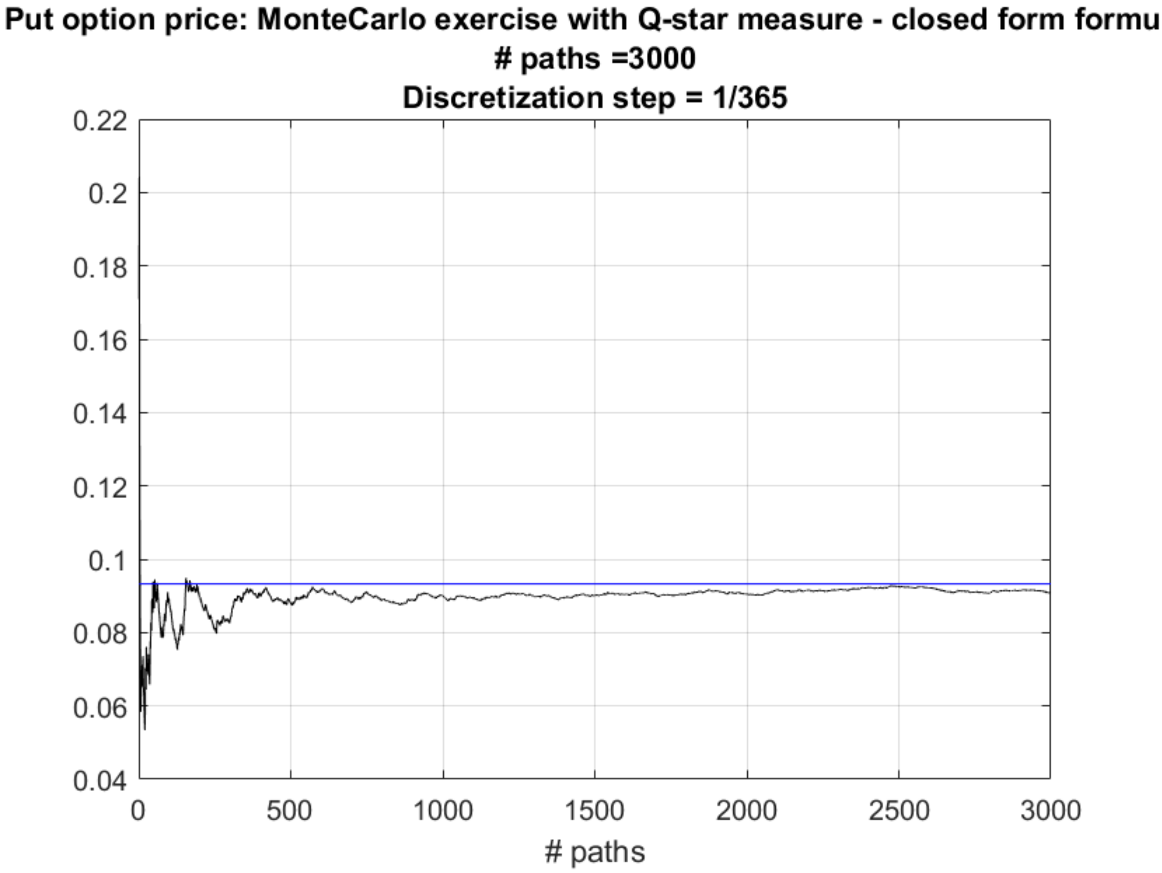
The Figure shows the convergence of the Monte Carlo swap price to the closed form swap price for a given choice of initial values of the state variables and maturity.

Figure 2: Call price



The Figure shows the convergence of the Monte Carlo call price to the closed form call price for a given choice of initial values of the state variables and maturity.

Figure 3: Put price



The Figure shows the convergence of the Monte Carlo put price to the closed form put price for a given choice of initial values of the state variables and maturity.

Table 1: Results of estimates

	parameter	std.err.	t-stat	p-value
λ	0.38631	0.00059497	649.3	0
β	0.03055	0.0000239	1278.33	0
s	0.00132	0.00000231	570.94	0
η	0.00829	0.00010077	82.22	0
α	0.00091	0.00000068	1338	0
σ	0.02059	0.00001018	2023.61	0
ξ	1.1408	0.0047767	238.83	0
μ	0.02264	0.00009816	230.66	0
κ	0.29929	0.00068047	439.83	0
ψ	1.0086	0.00058369	1727.91	0
ϕ	0.00118	0.00040851	2.89	0
θ	0.00001	0.00000523	1.57	0.06
δ	0.0003	0.0000109	27.86	0
Σ_1	0.00001	0.00000023	30.77	0
Σ_3	0	0.00000001	26.16	0
Σ_4	0	0.00000006	11.38	0
Σ_5	0	0.00000034	11.23	0
Σ_6	0.00001	0.00000023	38.73	0
Σ_7	0.00002	0.00000016	116.85	0
Σ_8	0.00003	0.00000015	176.89	0
Σ_9	0.00005	0.00000013	371.07	0
Σ_{10}	0.00006	0.00000012	463.98	0
Σ_{12}	0.00006	0.00000012	540.33	0
Σ_{15}	0.00007	0.00000012	595.35	0
Σ_{20}	0.00003	0.00000014	231.98	0
Σ_{25}	0.00001	0.00000021	63.94	0
Ψ_1	0	0.00000018	14.69	0
Ψ_2	0	0.00000004	31.04	0
Ψ_5	0	0	299.27	0
Ψ_7	0	0.00000001	224.97	0
Ψ_{10}	0	0.00000011	32.27	0
Ψ_{12}	0	0.00000018	25.02	0
Ψ_{15}	0	0.00000028	11.09	0
Ψ_{20}	0	0.00000037	9.23	0
Ψ_{30}	0.00001	0.00000073	7.1	0
Ω_1	0.00024	0.00000011	2131.08	0
Ω_2	0.00002	0.00000043	57.3	0
Ω_5	0	0.00000009	48.97	0

The Table reports the parameters of model (1, 2, 12) estimated using a quasi-Newton algorithm. The standard error (std.err.), the t-statistics (t-stat) and the p-value (p-value) are computed with the Huber sandwich estimator

Table 2: pricing error of inflation swaps

tenor	error	std	RMSE
1	7.50	14.36	16.20
2	0	0	0
3	-3.05	4.41	5.36
4	-4.53	6.71	8.09
5	-5.15	8.21	9.69
6	-5.10	8.79	10.16
7	-4.30	8.73	9.73
8	-2.96	8.23	8.75
9	-1.33	7.68	7.80
10	0.20	7.27	7.27
12	2.23	6.56	6.93
15	3.65	5.81	6.86
20	2.83	4.58	5.38
25	0.96	3.15	3.30
30	0	0	0

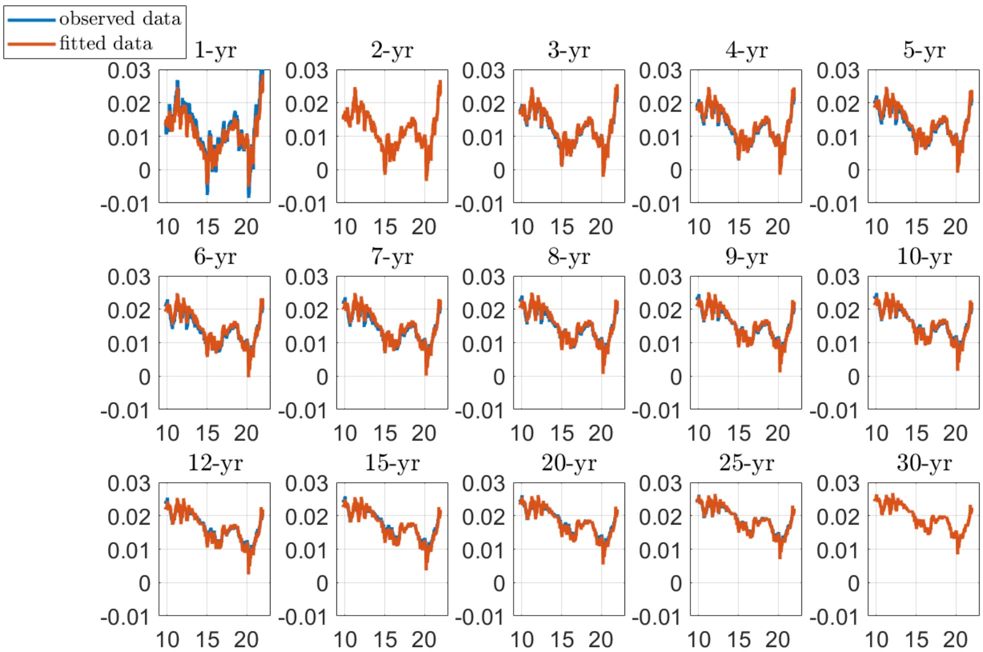
The Table reports the pricing error in basis points (error), the standard deviation (std.dev) and the root mean sqared erros (RMSE)in basis points of inflation swaps in equation (1).

Table 3: pricing error of implied variances

tenor	error	std.dev,	RMSE
1	-5.13	16.10	16.90
2	-1.84	10.67	10.82
3	0	0	0
5	-0.47	11.90	11.91
7	1.76	15.23	15.33
10	4.75	17.94	18.56
12	2.16	22.12	22.22
15	3.22	17.51	17.80
20	0.55	18.89	18.89
30	-5.42	21.86	22.52

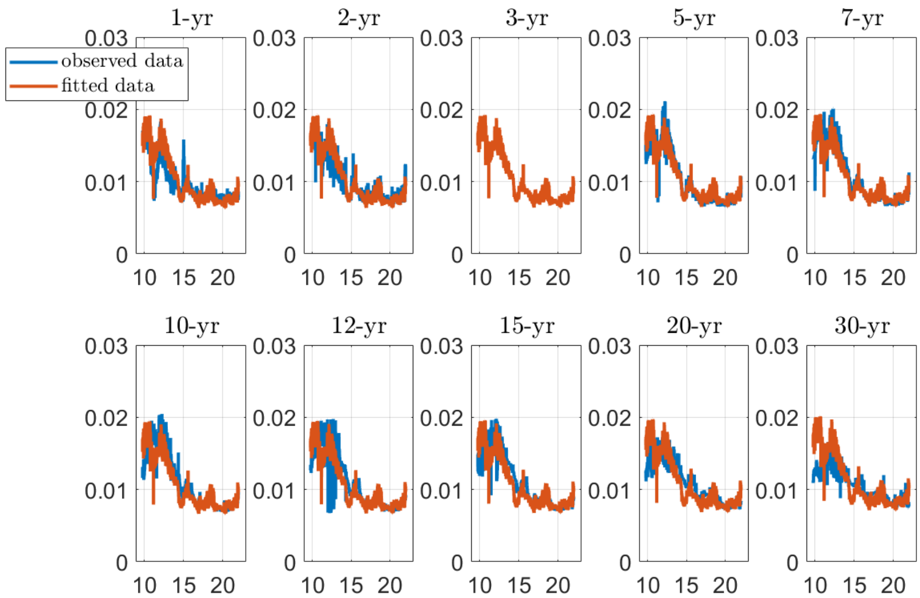
The Table reports the pricing errors in basis points (error), the standard deviation (std.dev) and the root mean sqared erros (RMSE) of implied volatility in equation (2).

Figure 4: observed and fitted inflation swap



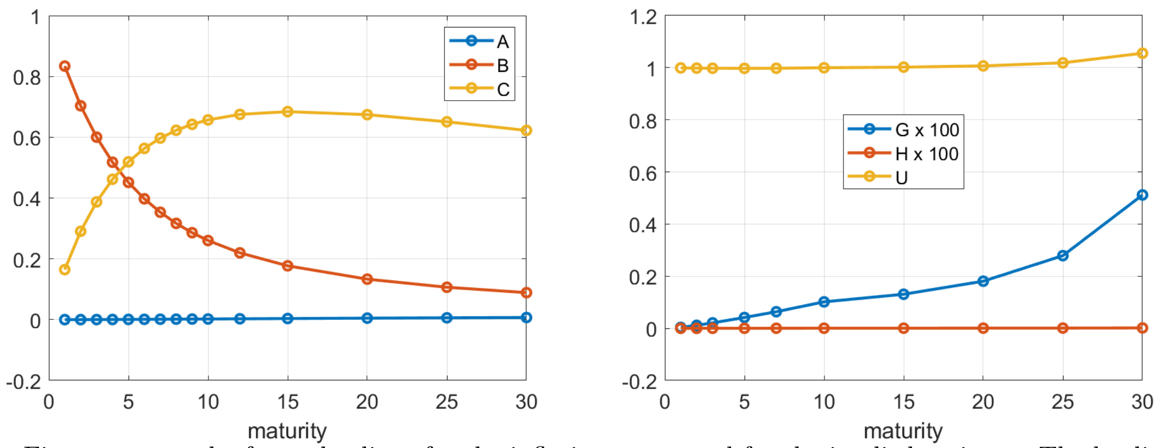
The Figure reports the observed and fitted inflation swaps under the $\mathbb{Q}$-measure.

Figure 5: observed and fitted standard deviation



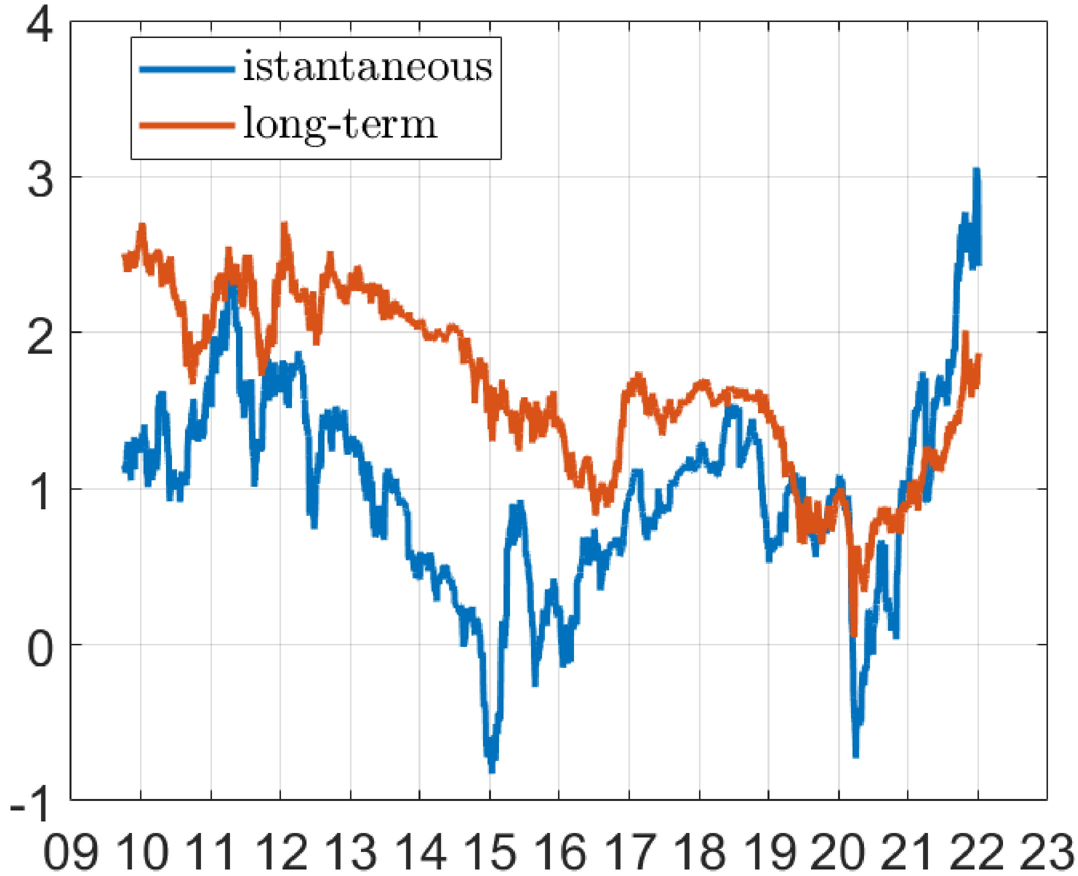
The Figure reports the observed and fitted standard deviation under the $\mathbb{Q}$-measure. The observed standard deviation is calculated as the second moment of the density implied in inflation caps and floors quotes.

Figure 6: factors of the inflation swaps term structure



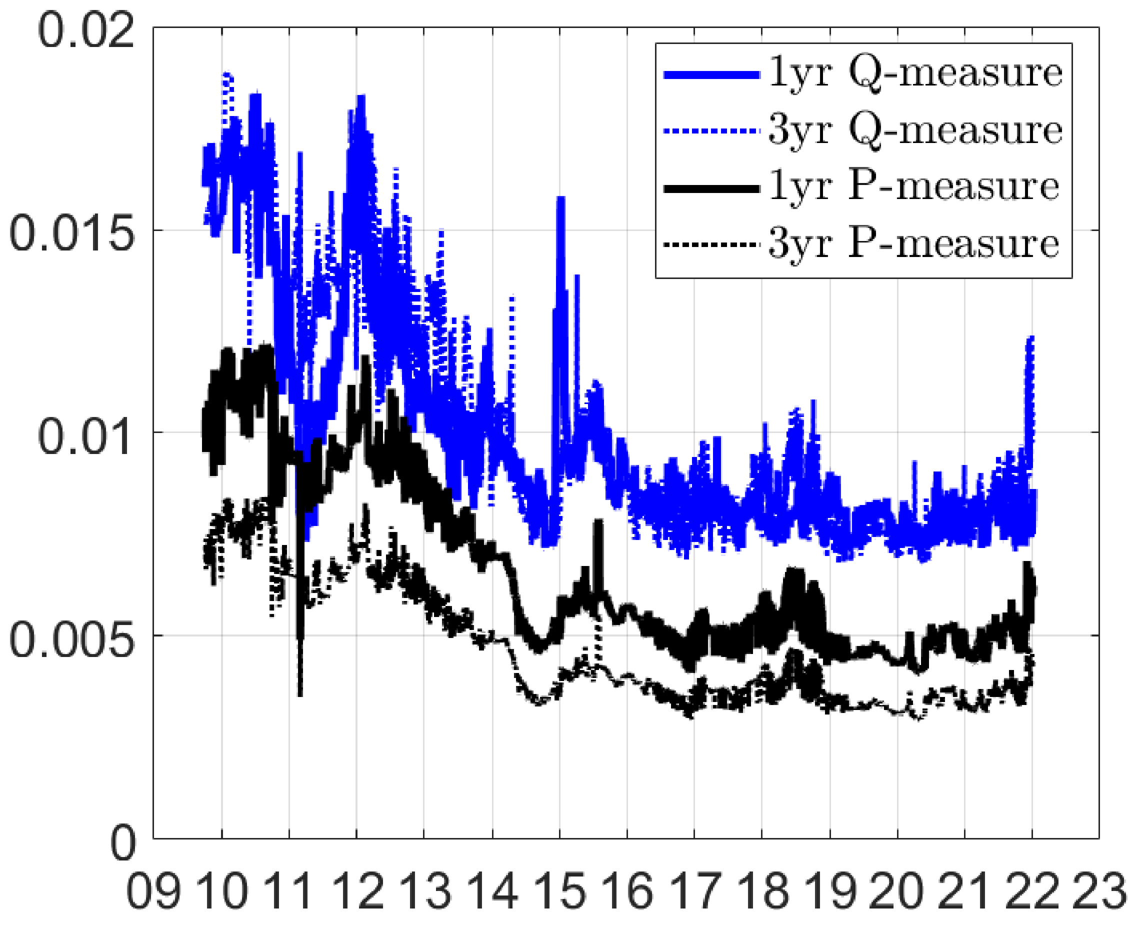
The Figure reports the factor loadings for the inflation swaps and for the implied variance. The loading of vectors G and H are multiplied by 100.

Figure 7: istantaneous and long-term inflation



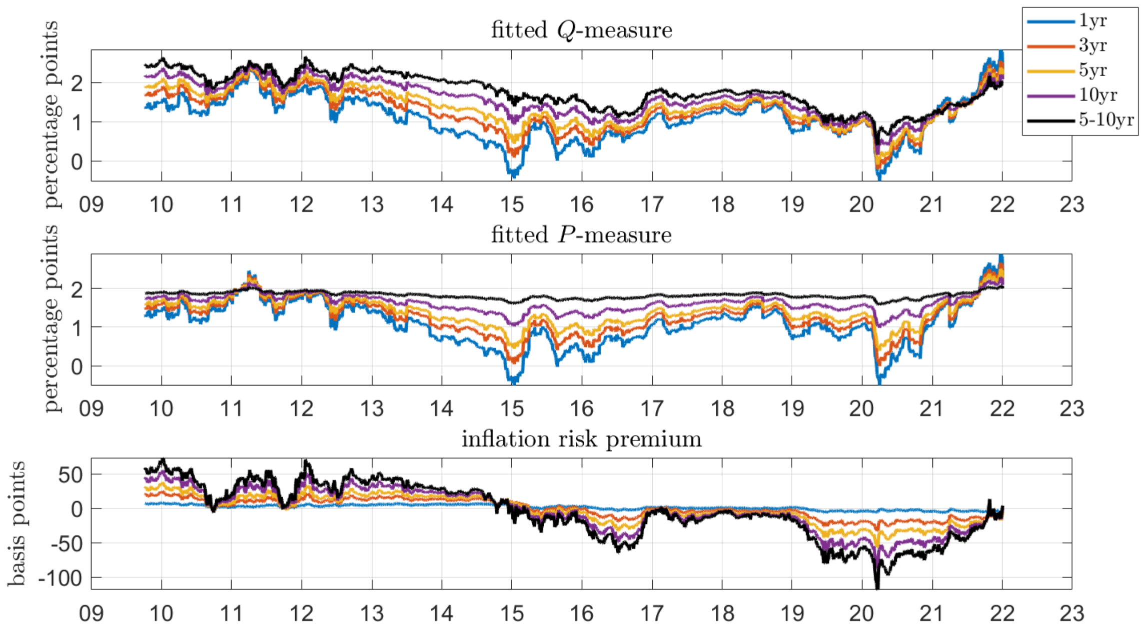
The Figure reports the istantaneous inflation (X) and long-term inflation (Y) estimated from the model.

Figure 8: implied variance under the $\mathbb{Q}$ and $\mathbb{P}$ measure



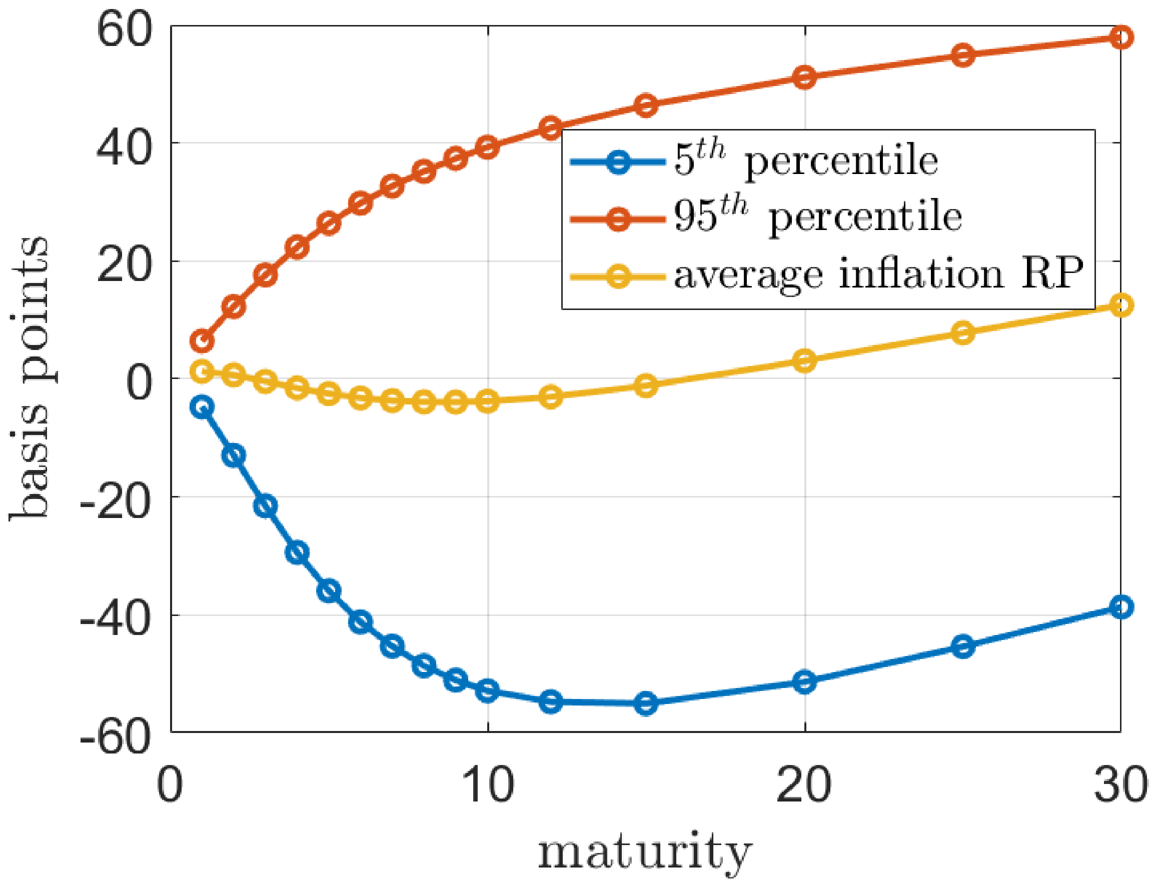
The Figure reports the fitted implied variance under the $\mathbb{Q}$ and the $\mathbb{P}$ measure.

Figure 9: fitted $\mathbb{Q}$-measure inflation swap (breakeven), fitted $\mathbb{P}$-measure inflation swap (expected) and inflation risk premium



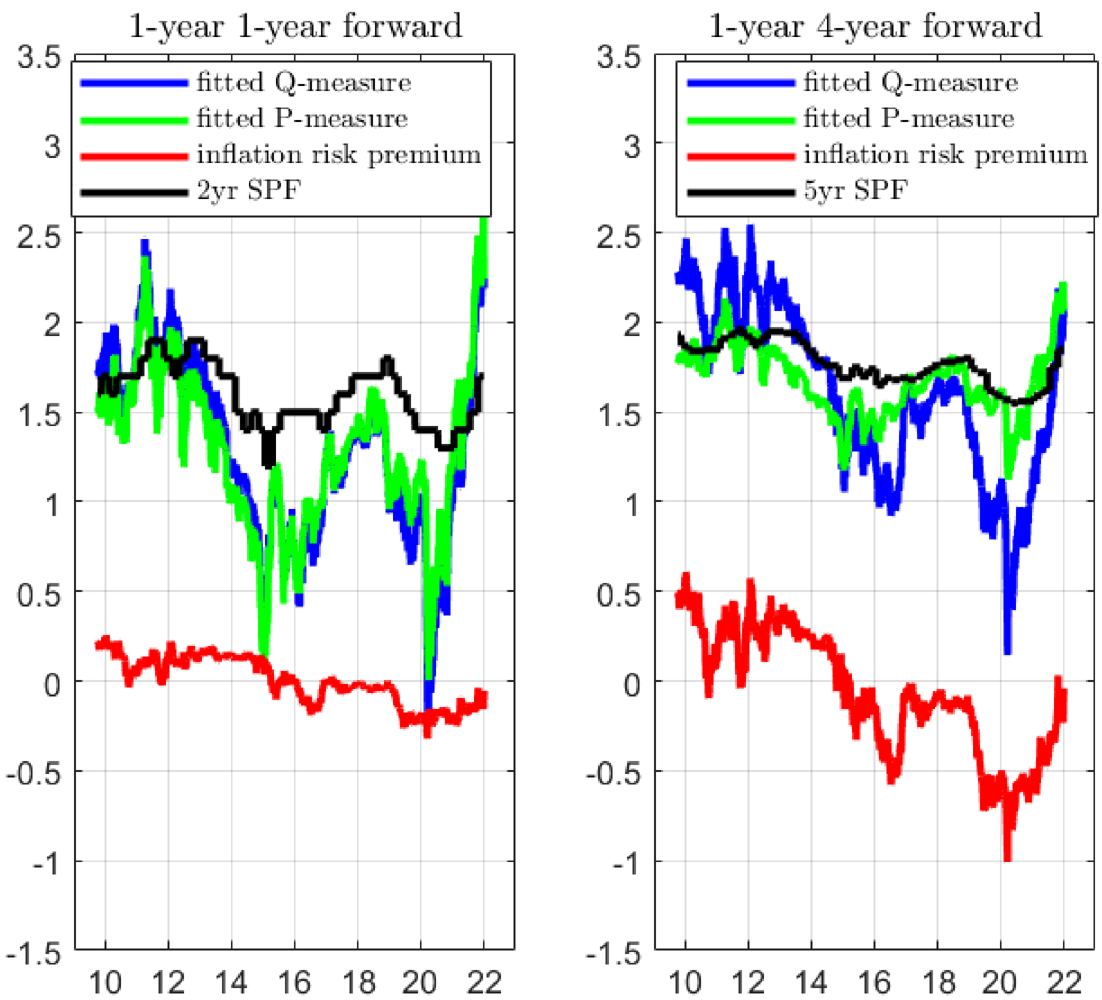
The Figure reports the fitted $\mathbb{Q}$-measure inflation swap (breakeven), the fitted $\mathbb{P}$-measure inflation swap (expected) and the inflation risk premium for the maturity 1-, 3-, 5-, 10-, and 5-year five year forward.

Figure 10: inflation risk premium



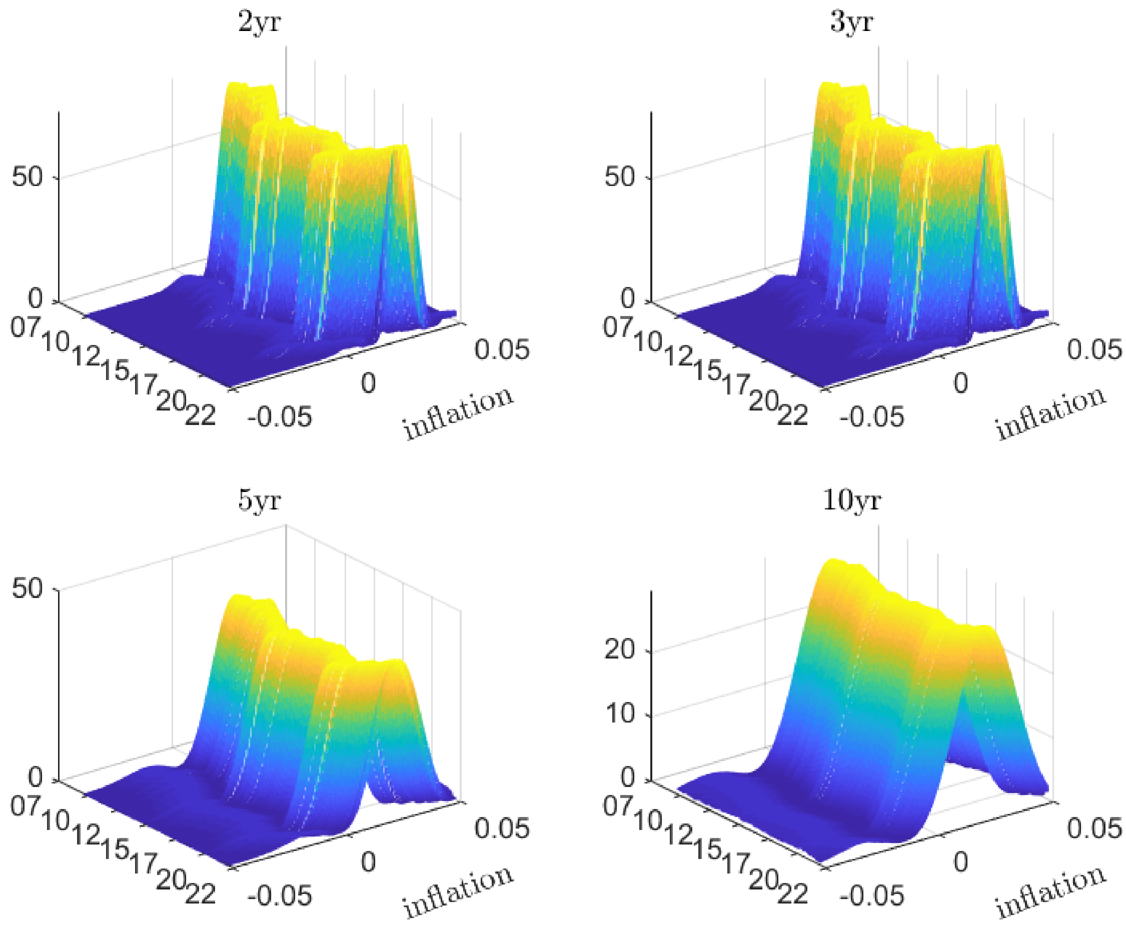
The Figure reports the average inflation riskpremium and the 5% and 95% percentiles.

Figure 11: fitted $\mathbb{Q}$-measure inflation, fitted $\mathbb{P}$-measure inflation, inflation risk premium and expected inflation surveyed by the SPF



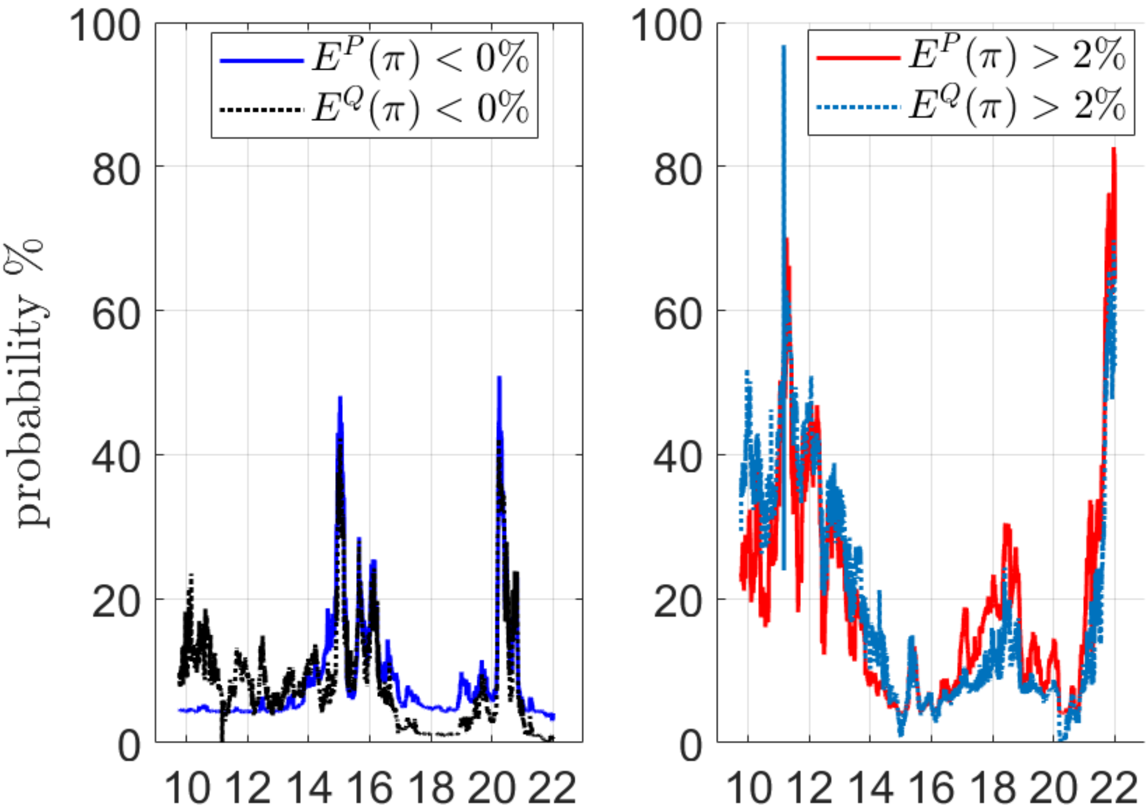
The Figure reports the 1-year forward fitted $\mathbb{P}$-measure (breakeven) inflation, the fitted $\mathbb{Q}$-measure (expected) inflation, the inflation risk premium, SPF annual inflation expected after one year (left panel) and the mean of the aggregate probability distribution of SPF annual inflation expected after 4 years (right panel).

Figure 12: $\mathbb{P}$-measure inflation densities



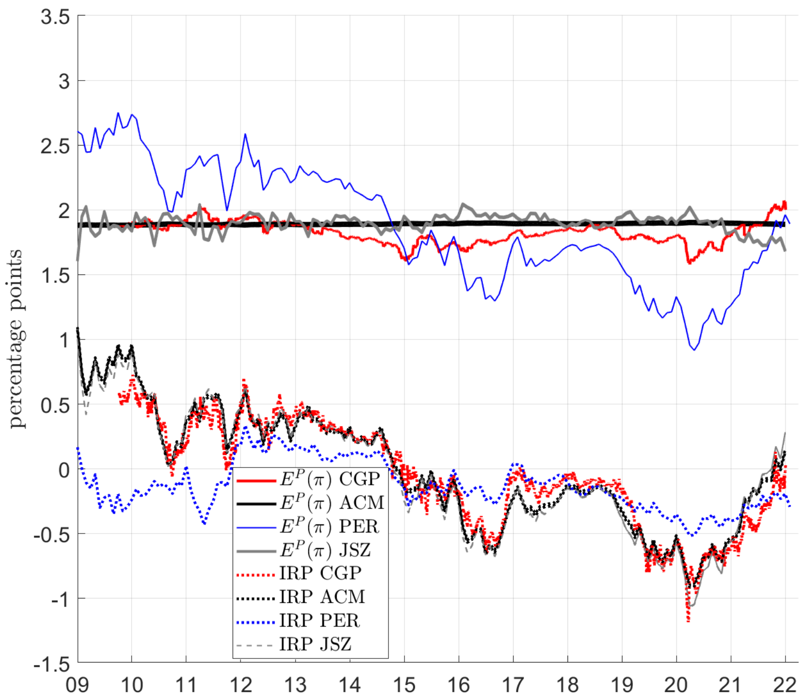
The Figure reports the densities of inflation under the $\mathbb{P}$ measure for the 2, 3, 5 and 10 year maturity.

Figure 13: probability of inflation lower than 0 and greater than 2% over the following 3 years



The Figure reports the $\mathbb{P}$-measure (–) and $\mathbb{Q}$-measure (- -) probability that inflation is lower than 0% and greater than 2% on average over the following three years.

Figure 14: 5-year five year forward expected inflation and inflation risk premium for different models



The Figure reports 5-year 5 year forward expected inflation and the inflation risk premium estimated by this paper (CGP), that estimated as in Adrian et al. (2013) (ACM), Joslin et al. (2011) (JSZ) and Pericoli (2019) (PER). JSZ and PER are at monthly frequency. The average of the long-term expected inflation estimated with the ACM and JSZ models is imposed to be equal to 1.9%, the long-term inflation aim of the ECB. The long-term expected inflation estimated with the CGP and PER models is unconstrained.