Strict Stationarity of INARCH(∞) Models

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Abstract: This paper establishes necessary and sufficient conditions for the existence of a strictly stationary solution for integer-valued autoregressive conditional heteroscedasticity (INARCH) processes. The results apply to INGARCH(p,q) and integrated INIGARCH(p,q) models, and its long-memory versions with hyperbolically decaying coefficients.

Keywords INARCH processes, Infinite matrices, Separable Hilbert space, Lyapunov exponent

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1 Introduction

In many real-world situations, we have to deal with non-negative integer-valued time series. Such time series are often produced in fields that include economics, insurance, medicine, epidemiology, queueing systems, communications, and meteorology and so on. Examples for the wide range of practical applications are the daily or monthly number of cases in epidemiology, the number of stock market transactions or stock price changes per minute in finance and the number of photon arrivals per microsecond measured in a biological experiment. Their analysis may present some difficulties, however, and if the analysis is based on stochastic models, these models have to reflect the integer peculiarity of the observed series. Various models have been suggested in the literature to tackle the problem of integer-valued time series analysis. These models include the traditional generalized linear model methodology and the state-of-the-art integer-valued autoregressive moving average (INARMA), and integer-valued generalized autoregressive conditional heteroscedasticity (INGARCH) processes. The first modeling approach is very simple and consists of choosing a suitable distribution for count data and an appropriate link function, (see Kedem and Fokianos, 2002). The second group of models are adaptation of the well-known ARMA and GARCH processes in the modeling of continuous-state and discrete-time series to count settings by means of thinning operators (see Weiß, 2008, for a recent review of the thinning operators). These processes are developed to model stationary count data. Therefore, considerable effort has been devoted to provide and prove general conditions that ensure existence and uniqueness of second-order stationary solutions using Hilbert space techniques (see Ferland et al., 2006; Latour, 1998; Doukhan and Wintenberger, 2008; Doukhan et al., 2012; Neumann, 2011). However, recent empirical observations indicate that some important count data in modeling are strictly stationary[2], and non square-integrable (see Segnon and Stapper, 2019).

The objective of this paper is to establish conditions for strict stationary of the INARCH processes. We make use of the multiplicative ergodic theorem developed by Ruelle (1982) for bounded operators in Hilbert space and show that the necessary and sufficient conditions for stationarity is the negativity of a Lyapunov exponent associated with these processes. Our result applies to the INGARCH model in Ferland et al. (2006), and INFIGARCH and INHYGARCH models in Segnon and Stapper (2019). Since the seminal paper by Bougerol and Picard (1992) the use of the multiplicative ergodic theorem to study the stationarity of ARCH-type processes has become very popular.

The rest of the paper is structured as follows. Section 2 briefly presents the INARCH processes. Our results are presented in Section 3 and Section 4 concludes.

[2]We recall, a process $\{X_t\}$ is strictly stationary if for all, $t, h \in \mathbb{Z}$, the law of $(X_t, X_{t+1}, \ldots, X_{t+h})$ is independent of t.

2 Poisson INARCH(∞) processes

2.1 Definition

A sequence of integer-valued random variables $\{Y_t\}_{t\in\mathbb{Z}}$ is said to be an INARCH(∞) process if:

(i) the distribution of Y_t conditional on the $\sigma-$field $\Omega_{t-1} = \sigma(Y_l, l \le t-1)$ is Poisson with mean λ_t,

(ii) there exist nonnegative constants c, ψ_i, $1 \le i \le \infty$, such that

$$\lambda_t = c + \psi(\mathrm{L})\, Y_t, \tag{1}$$

where $\Pr(\lambda_t > 0) = 1$ and $\psi(\mathrm{L}) = \sum_{i=1}^{\infty} \psi_i \mathrm{L}^i$.

This class of models also includes:

(a) The integer-valued HYGARCH(p, d, q) model for, c is an appropriately defined constant, and

$$\begin{aligned}\psi(\mathrm{L}) &= \left[1 - \frac{\Phi(\mathrm{L})\left(1 + \eta[(1-\mathrm{L})^d - 1]\right)}{\mathbf{B}(\mathrm{L})}\right] \\ &= \sum_{i=1}^{\infty} \psi_i \mathrm{L}^i,\end{aligned} \tag{2}$$

with $\beta_0 > 0$ and $\phi_1, \ldots, \phi_{m-1} \ge 0, \beta_1, \ldots, \beta_q \ge 0$, and $\psi_i \ge 0$ for all i. In Eq. (2), L denotes the lag operator. The lag polynomials are defined as $\Phi(\mathrm{L}) = [1 - \beta(\mathrm{L}) - \alpha(\mathrm{L})] = \sum_{i=1}^{m-1} \phi_i \mathrm{L}^i$, where $m = \max(p, q)$, $\alpha(\mathrm{L}) = \sum_{i=1}^{p} \alpha_i \mathrm{L}^i$, $\beta(\mathrm{L}) = \sum_{j=1}^{q} \beta_j \mathrm{L}^j$ and $\mathbf{B}(\mathrm{L}) = [1 - \beta(\mathrm{L})]$. $\eta \ge 0$ is an amplitude parameter, $d \in [0, 1]$ and $(1-\mathrm{L})^d$ is the fractional differencing operator given by

$$(1-\mathrm{L})^d = \sum_{k=0}^{\infty} \frac{\Gamma(k-d)\mathrm{L}^k}{\Gamma(-d)\Gamma(k+1)}, \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function.

(b) The integer-valued FIGARCH(p, d, q) model for $\eta = 1$ in Eq. 2.

(c) The integer-valued GARCH(p, q) model for $\eta = 0$ in Eq. 2.

Remark 1. *Segnon and Stapper (2019) show that for $\eta \in (0, 1)$ implies that $\psi(1) < 1$, and thus, the INHYGARCH process is covariance stationary.*

Remark 2. *Ferland et al. (2006) show that the INGARCH(p,q) process exists and is strictly stationary with finite first and second order moments, if and only if the following* *restriction is met:* $\sum_{i=1}^{p}\alpha_i+\sum_{j=1}^{q}\beta_j<1$*, which is equivalent to* $\sum_{i=1}^{\infty}\psi_i<1$*. In the simple INGARCH*$(1,1)$*,* $\psi_i=\alpha_1\beta_1^{i-1}$ *for* $i\geq 1$ *and the stationarity condition is well known to be* $\alpha_1+\beta_1<1$*, which is equivalent to* $\sum_{i=1}^{\infty}\alpha_1\beta_1^{i-1}<1$ *in the INARCH representation above. The INGARCH*$(1,1)$ *reduces to an integrated INGARCH*$(1,1)$ *when the sum of the lag coefficients is unity (*$\alpha_1+\beta_1=1$*). Segnon and Stapper (2019) point out that in the INFIGARCH(p,q)* $\sum_{i=1}^{\infty}\psi_i=1$*. Thus, the process is not covariance stationary. We note that the coefficient* ψ_i *in the INHYGARCH and INFIGARCH can be approximated by* ci^{-1-d}*, with c appropriately defined.*

Since the INFIGARCH(p,d,q) process is not covariance stationary, its appears that the INFIGARCH(p,d,q) is not a long memory model in the common sense. However, we aim to show in the next Section that the INARCH representation of the INFIGARCH(p,d,q) process is strictly stationary using a multiplicative ergodic theorem. Towards this end, we first look at the construction of an INARCH process.

2.2 Construction

Let $\{u_t\}_{t\in\mathbb{Z}}$ be a sequence of independent random variables with values in $\mathbb{N}$ ($\mathbb{N}$ is the set of non-negative integers) with common mean ω. For each $t\in\mathbb{Z}$ and $i\in\mathbb{N}$, let $\xi_t^{(i)}=\{\xi_{t,j}^{(i)}\}_{j\in\mathbb{N}}$ represent a sequence of independent random variables having a common mean ψ_i. All the variables u_s, $\xi_{t,j}^{(i)}$, ($s\in\mathbb{Z}, t\in\mathbb{Z}, i\in\mathbb{N}$ and $j\in\mathbb{N}$) are assumed to be mutually independent. Using these random variables, we introduce a sequence of random variables $\{Y_t^{(n)}\}$ that may be considered as successive approximations of Y_t:

$$Y_t^{(n)}=\begin{cases}0, & \text{if } n<0;\\ u_t, & \text{if } n=0;\\ u_t+\sum_{i=1}^{n}\sum_{j=1}^{Y_{t-i}^{(n-i)}}\xi_{t-i,j}^{(i)} & \text{if } n>0.\end{cases}\tag{4}$$

From (4) we can see that $Y_t^{(n)}$ is a finite sum of independent Poisson variables. So, the expectation and the variance of $Y_t^{(n)}$ are well defined. In the next Section we want to show that $Y_t^{(n)}$, as $n\to\infty$, admits an almost sure limit Y_t and that the limiting process $\{Y_t\}_{t\in\mathbb{Z}}$ satisfies (1).

3 Stationarity of INARCH(∞) processes

To prove the strict stationarity of $\{Y_t\}$ we first show that for any fixed n, $Y_t^{(n)}$ is strictly stationary.

3.1 Some basic definitions and results

Definition 1. *Let* $\{z_j\}_{j\in\mathbb{N}}$ *be a sequence of independent and identically distributed non-negative integer-valued random variables with mean* ψ *and finite variance* σ^2 *which is independent of a non-negative integer-valued random variable y. The generalized Steutel and van Harn operator,* $\psi\diamond$*, is defined as*

$$\psi \diamond y = \begin{cases} \sum_{i=1}^{y} z_i & \text{if } y > 0; \\ 0 & \text{if } y = 0. \end{cases} \tag{5}$$

Remark 3. *The sequence $\{z_j\}_{j\in\mathbb{N}}$ is called a counting sequence. Let $\alpha\diamond$ be another operator based on a counting sequence $\{x_j\}_{j\in\mathbb{N}}$. Both operators $\psi\diamond$ and $\alpha\diamond$ are said to be independent if and only if the counting sequences $\{z_j\}_{j\in\mathbb{N}}$ and $\{x_j\}_{j\in\mathbb{N}}$ are mutually independent.*

Using the operator from Eq. 5, we may rewrite the sequence of random variables $\{Y_t^{(n)}\}_{r\in\mathbb{N}}$ as

$$Y_t^{(n)} = \sum_{i=1}^{n} \mathbb{E}\left(\xi_{t-i}^{(i)}\right) \diamond Y_{t-i}^{(n-i)} + u_t, \quad n > 0, \tag{6}$$

where $\mathbb{E}\left(\xi_{t-i}^{(i)}\right) = \psi_i$.

Proposition 1. *If $\psi(1) < 1$ then the sequence $\{Y_t^{(n)}\}_{n\in\mathbb{N}}$ has an almost sure limit.*

Proof. We closely follow the Proof of Proposition 2 in Ferland et al. (2006), Page 928. It follows from Eq. (4) that $Y_t^{(n)}$ is obtained through a cascade of thinning operations along the sequence $\{u_t\}_{t\in\mathbb{Z}}$. So, the expectation and the variance of $Y_t^{(n)}$ are well defined and given by

$$\begin{aligned} \mu_n &= \mathbb{E}\left(u_t + \sum_{i=1}^{n} \sum_{j=1}^{Y_{t-i}^{(n-i)}} \xi_{t-i,j}^{(i)}\right) \\ &= \omega + \sum_{i=1}^{n} \mathbb{E}\left(\sum_{j=1}^{Y_{t-i}^{(n-i)}} \xi_{t-i,j}^{(i)}\right) \end{aligned} \tag{7}$$

Let $(\Omega, \mathfrak{F}, P)$ be the common probability space on which the relevant random variables are defined. Because Y_j^n is a non-decreasing sequence of non-negative integers, we have

$$\forall \omega \in \Omega, \lim_{n\to\infty} Y_t^n(\omega) = Y_t \tag{8}$$

which is either finite or infinite. We will show that the set

$$A_\infty = \{\omega : Y_t(\omega) = \infty\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \limsup_n A_n \tag{9}$$

is of probability zero, where

$$A_n = \left\{\omega : Y_t^n(\omega) - Y_t^{n-1}(\omega) > 0\right\}, \text{ for } n > 1. \tag{10}$$

On the one hand:

$$\mathbb{E}\left(Y_t^n - Y_t^{n-1}\right) \geq \sum_{k=1}^{\infty} Pr\left\{\omega : Y_t^n(\omega) - Y_t^{n-1}(\omega) = k\right\} = Pr(A_n). \tag{11}$$

On the other hand:

$$\mathbb{E}\left(Y_t^n - Y_t^{n-1}\right) = \mu_n - \mu_{n-1} \equiv \nu_n. \tag{12}$$

Obviously, the sequence ■

Proposition 2. *Let $\mathbf{C}_t = \{c_{i,j}\}_{1<i,j<n}$ be an finite-dimensional random matrix given by*

$$\mathbf{C}_t = \begin{pmatrix} \xi_{t-1}^{(1)} & \xi_{t-2}^{(2)} & \dots & \xi_{t-n}^{(n)} \\ 1 & 0 & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \dots & 1 & 0 \end{pmatrix}, \tag{13}$$

$\mathbf{Z}_{t+1} = \left(Y_t^{(n)}, Y_{t-1}^{(n-1)}, Y_{t-2}^{(n-2)}, \dots Y_{t-n}^{(0)}\right)'$ *and* $\mathbf{U}_{t+1} = (u_t, 0, \dots)'$.
Then, (6) has a stationary and ergodic solution if and only if

$$\mathbf{Z}_{t+1} = \mathbb{E}\left(\mathbf{C}_{t+1}\right) \diamond \mathbf{Z}_t + \mathbf{U}_{t+1}, \quad t \in \mathbb{Z}, \tag{14}$$

has a stationary and ergodic solution where

$$\mathbb{E}\left(\mathbf{C}_t\right) \diamond := \tilde{c}_{ij} \diamond = \begin{cases} \psi \diamond & i = 1; \\ 1 \diamond & i = j + 1 \\ 0 \diamond & \text{otherwise}. \end{cases}$$

Proof. Eq. (14) is a state-space representation of (6), and thus, any stationary solution of (14) is also a stationary solution of (6) and vice versa. Analogously, any ergodic solution of (14) is also an ergodic solution of (6), and vice versa. The proof of the ergodicity follows from Lemma A 1.2.7 in Brandt et al. (1990).

■

Lemma 1. *Let $\psi(z) = z^n - \alpha_1 z^{n-1} - \cdots - \alpha_{n-1} z - \alpha_n$ with $\sum_{k=1}^n |\alpha_k| \leq 1$ and $\alpha_n > 0$. Then the roots of $\psi(z)$ are all inside the unit circle.*

Proof. Let us consider the unit circle $\zeta = \{z : |z| = 1\}$ and suppose $\sum_{k=1}^n |\alpha_k| < 1$. The functions $h(z) = z^n$ and $T(z) = -(\alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \cdots + \alpha_n)$ are both analytic inside and on ζ. Hence, on ζ,

$$|T| \leq \sum_{k=0}^{r-1} |\alpha_{n-k} z^k| \leq \sum_{k=0}^{n-1} |\alpha_{n-k}| < 1 = |h|.$$

Based on the theorem of Rouché, $h(z)$ and $h(z) + T(z)$ have the same number of zeros inside ζ. But h has n zeros inside ζ. Therefore, we conclude that all roots of $\alpha(z)$ are inside the unit circle. ■

Lemma 2. *(Lemma 2.1 in Bougerol and Picard (1992)) Let* $\{\mathbf{A}_n, n \in \mathbb{Z}\}$ *be a sequence of independent, identically distributed, random matrices such that* $\mathbb{E}(\log^+ \|\mathbf{A}_0\|)$ *is finite. If, almost surely,*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\mathbf{A}_0\mathbf{A}_{-1}\cdots\mathbf{A}_{-n}\| = 0,$$

then the top Lyapunov exponent associated with this sequence is strictly negative.

Proposition 3. *The process defined in Eq. (14) has a unique strictly stationary and ergodic solution if and only if the top Lyapunov exponent* γ *associated with the random matrices* $\{\mathbf{C}_t\}_{t\in\mathbb{Z}}$ *is strictly negative. The unique strictly stationary solution* $(\mathbf{Y}_t)_{t\in\mathbb{Z}}$ *of (14) is given by*

$$\mathbf{Z}_t = \sum_{k=0}^{\infty} \mathbb{E}(\mathbf{C}_t\mathbf{C}_{t-1}\ldots\mathbf{C}_{t-k+1}) \diamond \mathbf{U}_{t-k}. \tag{15}$$

Proof. We can see that the random matrices $\{\mathbf{C}_t\}$ in Eq. 13 consist of independent and identically distributed non-negative integer-valued random variables, $\xi_t^{(i)}$, with the baseline distribution f (Poisson or negative Binomial) and with a finite mean, ψ_i and variance. This means that all the coefficients of these matrices are integrable. Furthermore, the random vectors $\{\mathbf{U}_t\}_{t\in\mathbb{Z}}$ contain *i.i.d.* non-negative integer-valued random variables and therefore are also integrable. All these imply that $\mathbb{E}(\log^+\|\mathbf{C}_0\|)$ and $\mathbb{E}(\log^+\|\mathbf{U}_0\|)$ are finite and therefore, the top Lyapunov exponent γ of the sequence $\{\mathbf{C}_t, t\in\mathbb{Z}\}$ is well defined.

Suppose that there exists a strictly stationary solution $\{\mathbf{Z}_t\}_{t\in\mathbb{Z}}$ of Eq. (6). By iterating Eq. (14), we have for $t > 0$,

$$\begin{aligned}
\mathbf{Z}_0 &= \mathbb{E}(\mathbf{C}_0) \diamond \mathbf{Z}_{-1} + \mathbf{U}_0 \\
&= \mathbb{E}(\mathbf{C}_0\mathbf{C}_{-1}) \diamond \mathbf{Z}_{-2} + \mathbf{U}_0 + \mathbb{E}(\mathbf{C}_0) \diamond \mathbf{U}_{-1} \\
&= \mathbb{E}(\mathbf{C}_0\mathbf{C}_{-1}\ldots\mathbf{C}_{-t}) \diamond \mathbf{Z}_{-t-1} + \mathbf{U}_0 + \sum_{j=1}^{t} \mathbb{E}\left(\mathbf{C}_0\ldots\mathbf{C}_{-j+1}\right) \diamond \mathbf{U}_{-j} \\
&= \mathbb{E}\left(\prod_{j=0}^{t} \mathbf{C}_{-j}\right) \diamond \mathbf{Z}_{-t-1} + \sum_{j=0}^{t} \mathbb{E}\left(\prod_{i=0}^{j-1} \mathbf{C}_{-i}\right) \diamond \mathbf{U}_{-j} \\
\mathbf{Z}_0 &= \mathbb{E}\left(\mathbf{C}^{(t)}\right) \diamond \mathbf{Z}_{-t-1} + \mathbf{U}^{(t)}
\end{aligned}$$

with $t \in \mathbb{N}_0$ and where $\prod_{i=0}^{-1} \mathbf{C}_{-i} = 1$.

All the coefficients of $\mathbf{C}_t$, $\mathbf{Y}_t$ and $\mathbf{U}_t$ are nonnegative. The characteristic polynomial of $\mathbb{E}(\mathbf{C}_t)$ is $\Psi(z) = z^n - \psi_1 z^{n-1} - \cdots - \psi_{n-1}z - \psi_n$. By Lemma 1, the roots of $\Psi(z)$ are all inside the unit circle, then $\lim_{t\to\infty} \mathbb{E}\left(\left(\prod_{j=0}^{t} \mathbf{C}_{-j}\right) e_i\right) = 0$ a.s. where e_i denotes the canonical basis of $\mathbb{R}^n$.

$$\lim_{t\to\infty} \mathbb{E}\left(\prod_{j=0}^{t} \mathbf{C}_{-j}\right) = 0$$

$$\lim_{t\to\infty} \mathbb{E}\left(\|\prod_{j=0}^{t} \mathbf{C}_{-j}\|\right) = 0$$

then $\lim_{t\to\infty} \mathbb{E}\left(\prod_{j=0}^{t} \mathbf{C}_{-j}\right) \diamond \mathbf{Y}_{-t-1} = 0$ a.s. According to Lemma 2 the associated top Lyapunov exponent γ is strictly negative, so that the series $\sum_{j=0}^{t} \mathbb{E}\left(\prod_{i=0}^{j-1} \mathbf{C}_{-i}\right) \diamond \mathbf{U}_{-j}$ converges a.s. Therefore, $\{\mathbf{Z}_t\}_{t\in\mathbb{Z}}$ is a strictly stationary and ergodic process, solution of Eq. (6).

Now, we aim to prove the uniqueness of the strictly stationary solution. Let $\{\mathbf{W}_t\}_{t\in\mathbb{Z}}$ be another strictly stationarity solution of Eq. (14). The norm of the following difference for $t > 0$

$$\begin{aligned} \|\mathbf{Z}_0 - \mathbf{W}_0\| &= \|\,\mathbb{E}\,(\mathbf{C}_0\mathbf{C}_{-1}\ldots\mathbf{C}_{-t}) \diamond (\mathbf{Z}_{-t-1} - \mathbf{W}_{-t-1})\,\| \\ &\le \|\,\mathbb{E}\,(\mathbf{C}_0\mathbf{C}_{-1}\ldots\mathbf{C}_{-t}) \diamond \|\,\|\mathbf{Z}_{-t-1} - \mathbf{W}_{-t-1}\| \\ &\le \|\,\mathbb{E}\left(\mathbf{C}^{(t)}\right) \diamond \|\,\|\mathbf{Z}_{-t-1} - \mathbf{W}_{-t-1}\|, \end{aligned}$$

by Lemma 1, converges to 0, a.s. and the fact that the law of the difference $(\mathbf{Z}_{-t-1} - \mathbf{W}_{-t-1})$ is independent of t, imply that $\mathbf{Z}_0 - \mathbf{W}_0$ converges to 0 in probability. We conclude that $\mathbf{Z}_0 = \mathbf{W}_0$ and that Eq. (6) has a unique solution, once the counting process are known. ∎

Corollary 1. *The process $\{Y_t\}_{t\in\mathbb{Z}}$ is a strictly stationary and ergodic process.*

The Proposition 3 holds its validity for any fixed number n. When $n \to \infty$, then $\mathbf{C}_t$ becomes an infinite dimensional random matrix and as pointed out by Schaumlöffel (1991) the multiplicative ergodic theorem of Oseledec (1968) cannot easily be extended to an infinite-dimensional context. The reason is that in infinite dimensions the orbits of a linear operator can be quite complicated. For that reason we need here additional assumptions to guarantee the validity of the Proposition 3. Following Ruelle (1982) we show that the compactness of the linear operator $\mathbf{C}$ is the necessary and sufficient condition for the strict negativity of the associated top Lyapunov exponent.

Let H denotes a separable infinite-dimensional Hilbert space and $B(H)$ the algebra of all bounded operators and the ideal of all compact operators. The space $H^{\wedge q}$ is the q^{th} exterior power of H and it consists of the completely antisymmetric elements of the Hilbert space tensor product of q copies of H. Let $\{e_n\}$ be any orthonormal basis $\mathcal{B}$ for H. According to Definition 7 in Van Barel et al. (1999) [Definition of extended infinite companion matrix], the $\infty \otimes \infty$ matrix $\mathbf{C} = (c_{ij})$, $c_{ij} = (\mathbf{C}e_j|e_i)$, $i, j \in \mathbb{N}$, see Eq. 13, representation of the operator $\mathbf{C}$ has a block structure that corresponds to the block structure of the basis $\mathcal{B}$. Formally, we have

$$\mathbf{C} = \left[\mathbf{C}_{i,j}\right]_{i,j=1}^{\infty}, \tag{16}$$

where the blocks $C_{i,j}$ are square of order K.

To prove the compactness of $\mathbf{C}$ we need first to prove that the linear bounded operator $\mathbf{C}$ has a tri-block diagonal matrix representation with finite blocks. This idea has been put forward by Bakić and Guljaš (1999) and the following Proposition shows that a tri-block diagonal matrix representation for the bounded operator $\mathbf{C}$ can be obtained from any orthonormal basis of the Hilbert space H by an arbitrary small Hilbert-Schmidt perturbation.

Proposition 4. *Suppose* $\mathbf{C} \in B(H)$ *be an operator having a tri-block diagonal matrix*

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_{1,1} & \mathbf{C}_{1,2} & 0 & & & & \\ \mathbf{C}_{2,1} & \mathbf{C}_{2,2} & \mathbf{C}_{2,3} & 0 & & & \\ 0 & \mathbf{C}_{3,2} & \ddots & \ddots & \ddots & & \\ & 0 & \ddots & \mathbf{C}_{n,n} & \mathbf{C}_{n,n+1} & 0 & \\ & & \ddots & \mathbf{C}_{n+1,n} & \mathbf{C}_{n+1,n+1} & \ddots & \ddots \\ & & & 0 & \ddots & \ddots & \\ & & & & \ddots & & \end{pmatrix}, \tag{17}$$

according to the decomposition $H = \oplus_{n=1}^{\infty} H_n$ *onto finite square dimensional subspaces* H_n *and*

$$\lim_n \|\mathbf{C}_{n,n}\| = \lim_n \|\mathbf{C}_{n+1,n}\| = \lim_n \|\mathbf{C}_{n,n+1}\| = 0. \tag{18}$$

Then $\mathbf{C}$ *is a compact operator.*

Proof. Given any orthonormal basis in H and according to Lemma 2 in Bakić and Guljaš (1999) $\mathbf{C}$ allows a suitable Hilbert-Schmidt perturbation and thus, has a tri-block matrix representation. With the tri-block matrix representation of $\mathbf{C}$ the proof of the proposition follows from Theorem 2 in Bakić and Guljaš (1999).

■

The following assumptions are provided in Ruelle (1982). Let (M, Ω, p) be a probability space and $\vartheta : M \to M$ a measurable p−preserving transformation on H. Let $\mathbf{C} : \Omega \to \mathfrak{L}(H)$ be measurable to the bounded operators such that

(1.1) $\log^+ \|\mathbf{C}(\cdot)\| \in L^1(M, p)$.

Let us define

$$\mathbf{C}_x^t = \mathbf{C}(\vartheta^{t-1}x) \cdots \mathbf{C}(\vartheta x)\mathbf{C}(x).$$

Then, there exists a subset $\Gamma^+ \subset M$ such that $\vartheta\Gamma^+ \subset \Gamma^+$, $p(\Gamma^+) = 1$ and

(1.2) $\limsup_{t\to\infty} \log \|\mathbf{C}(\vartheta^{t-1}x)\| \le 0$, if $x \in \Gamma^+$.

(1.3) Furthermore, there exist ϑ−invariant functions $l_q^+ : \Gamma^+ \to \mathbb{R} \bigcup \{-\infty\}$ such that

$$\lim \frac{1}{t} \log \|(\mathbf{C}_x^t)^{\wedge q}\| = l_q^+$$

if $x \in \Gamma^+$, for all integers $q > 0$.

Proposition 5. *Suppose that the assumptions (1.1), (1.2) and (1.3) hold and that* $\mathbf{C}$ *is compact. Then the top Lyapunov exponent* γ *associated with the infinite random matrices*

$\{\mathbf{C}_t\}_{t\in\mathbb{Z}}$ *is strictly negative* $(-\infty)$ *and the process defined in Eq. (14) has a unique strictly stationary and ergodic solution,* $(\mathbf{Z}_t)_{t\in\mathbb{Z}}$*, that is given by*

$$\mathbf{Z}_t = \sum_{k=0}^{\infty} \mathbb{E}\left(\mathbf{C}_t \mathbf{C}_{t-1} \ldots \mathbf{C}_{t-k+1}\right) \diamond \mathbf{U}_{t-k}. \tag{19}$$

Proof. The assumptions (1.1) and (1.2) follow from the square integrability of $\log^+ \|\mathbf{C}\|$ and the ergodic theorem. For $q = 1$, $\lim \frac{1}{t} \log \|(\mathbf{C}^t)\| = l_1$. $\mathbf{C}^t$ acts as a bounded linear operator on $H^{\wedge q}$ and since $\|(\mathbf{C}^t)^{\wedge q}\| \leq \|\mathbf{C}^t\|^q$, the assumption (1.3) holds.

Let define

$$\varpi_N = \{x \in \Gamma^+ : \lim_{q\to\infty} \frac{1}{q} l_q^+(x) \geq -N\}.$$

Then

$$\begin{aligned} -Np(\varpi_N) &\leq \int_{\varpi_N} p(dx) \frac{1}{q} l_q^+(x) \\ &\leq \int_{\varpi_N} p(dx) \frac{1}{q} \log \|\mathbf{C}(x)^{\wedge q}\|. \end{aligned} \tag{20}$$

Because $\mathbf{C}(x)$ is compact, it follows that when $q \to \infty$, then $\frac{1}{q} \log \|\mathbf{C}(x)^{\wedge q}\| \to -\infty$. Since

$$\frac{1}{q} \log \|\mathbf{C}(\cdot)^{\wedge q}\| \leq \log^+ \|\mathbf{C}(\cdot)\| \in L^1(M, p),$$

we must have $p(\varpi_N) = 0$ for all real N.

■

Corollary 2. *Let assume that the support of f is unbounded, $f(\{0\}) = 0$ and all the coefficients ψ are nonnegative. Then, if $\sum_{i=1}^n \psi_i = 1$, then the INARCH process defined in Eq. (14) has a unique stationary solution.*

Proof. By induction on n, we have

$$\det\left(zIn - \mathbb{E}(\mathbf{C}_1)\right) = z^n \left(1 - \sum_{i=1}^{n} \psi_i z^{-i}\right).$$

The inequality $|a - b| \geq |(|a| - |b|)|$ implies that if $|z| > 1$, then

$$\det\left(zI_n - \mathbb{E}(\mathbf{C}_1)\right) > 1 - \sum_{i=1}^{n} \psi_i. \tag{21}$$

Since the right-hand side is zero and since $\det(zI_n - \mathbb{E}(\mathbf{C}_1)) = 0$, we conclude that the spectral radius ρ of the matrix $\mathbb{E}(\mathbf{C}_1)$ is 1. Furthermore, all the coefficients of the matrix $\mathbf{C}_2\mathbf{C}_1$ are almost surely positive and $\mathbf{C}_1$ has no zero column nor zero row. Since $\mathbf{C}_1$ is not a.s. bounded, these properties imply by (Kesten and Spitzer, 1984, theorem 2) that the top Lyapunov exponent γ satisfies $\gamma < \log \rho$. As result, $\gamma < 0$ and the corollary follows from Theorem 3. ■

4 Conclusion

This paper has provided general conditions for the existence and uniqueness of a strictly stationary solution independent of future for the INARCH(∞) processes. Using the theory of products of infinite random matrices in a separable Hilbert space we show that the INGARCH process and its long memory versions recently developed in Segnon and Stapper (2019), in particularly the INFIGARCH process, are strictly stationary.

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