

Predictable Forward Performance Processes: Infrequent Evaluation and Robo-Advising Applications*

Gechun Liang[†]

Moris S. Strub[‡]

Yuwei Wang[§]

Abstract

We study discrete-time predictable forward processes when trading times do not coincide with performance evaluation times in the binomial tree model for the financial market. The key step in the construction of these processes is to solve a linear functional equation of higher order associated with the inverse problem driving the evolution of the predictable forward process. We provide sufficient conditions for the existence and uniqueness and an explicit construction of the predictable forward process under these conditions. Furthermore, we show that these processes are time-monotone in the evaluation period. Finally, we argue that predictable forward preferences are a viable framework to model preferences for robo-advising applications and determine an optimal interaction schedule between client and robo-advisor that balances a tradeoff between increasing uncertainty about the client's beliefs on the financial market and an interaction cost.

Keywords — forward performance processes, robo-advising, binomial tree model, portfolio selection, functional equation

*Author order is alphabetical. All authors are co-first authors of this paper. Helpful comments and suggestions from Martin Herdegen is gratefully acknowledged. Moris Strub gratefully acknowledges funding through the National Natural Science Foundation of China under Grant No. 72050410356.

[†]Department of Statistics, University of Warwick. Email: g.liang@warwick.ac.uk

[‡]Department of Information Systems and Management Engineering, Southern University of Science and Technology. Email: strub@sustech.edu.cn

[§]Corresponding author. Department of Statistics, University of Warwick and Department of Information Systems and Management Engineering, Southern University of Science and Technology. Email: yuwei.wang.2@warwick.ac.uk

1 Introduction

Classical expected utility maximization requires to determine ex ante three basic elements: the investment horizon, the market model, and the performance criterion in terms of a utility function applying at the chosen terminal time. This fundamental setup has, however, two important limitations. First, the investor must pre-specify her future risk preference for evaluating the performance of investment strategies and the market model for describing asset dynamics for the entire investment horizon. As a consequence, the risk preference and the market model cannot be adjusted to new market observations over time. This is problematic, especially when the investment horizon lies in the distant future. Second, the investment horizon needs to be set before the investor enters the market.

Forward performance processes are an alternative performance criterion that can address these issues. Their continuous-time version was introduced by Musiela and Zariphopoulou in [Musiela and Zariphopoulou \(2006, 2008, 2009, 2010\)](#), and further developed in, for example, [Chong \(2019\)](#); [He et al. \(2021\)](#); [Henderson and Hobson \(2007\)](#); [Hu et al. \(2020\)](#); [Källblad \(2020\)](#); [Källblad et al. \(2018\)](#); [Liang and Zariphopoulou \(2017\)](#); [Nadtochiy and Tehranchi \(2017\)](#); [Shkolnikov et al. \(2016\)](#); [Žitković \(2009\)](#). In contrast, the discrete-time case is less well understood. To the best of our knowledge, the only two studies so far concerned with the analysis thereof are [Angoshtari et al. \(2020\)](#), where the framework was first introduced, and [Strub and Zhou \(2021\)](#) who extend some of the key results therein to more general models for the financial market and investigate the associated dynamics of risk preferences. An advantage of the discrete-time formulation of forward performance processes is that those are *predictable* instead of just adapted. This leads to a more intuitive relation of the utility functions at two consecutive time points. We herein build on the work of [Angoshtari et al. \(2020\)](#), and aim to extend their key results to the multi-period binomial tree model for the financial market. A key feature, both conceptually and technically, of this extension is that performance evaluation times generally do not coincide with trading times, but occur at a lower frequency. This setting is of particular relevance for wealth management, where interaction with the client often occurs

at a lower frequency than trading.

According to the general scheme developed in [Angoshtari et al. \(2020\)](#), the key step in the construction of a predictable forward process is to solve an associated *inverse* investment problem, where one is given an initial utility function and model for the market and seeks to determine a utility function applying at terminal time such that the initial utility function becomes the value function of the resulting expected utility maximization problem. Whereas this is a single-period problem in the binomial case studied in [Angoshtari et al. \(2020\)](#), we herein face a multi-period inverse investment problem. Because the financial market is complete, the results of [Strub and Zhou \(2021\)](#) apply and a solution to the multi-period inverse investment problem can be obtained by solving an associated generalized integral equation. In the binomial tree model considered herein, the associated generalized integral equation is a linear functional equation of higher order. Our main technical contributions are sufficient conditions for existence and uniqueness for the associated equation as well as an explicit construction of a solution under those conditions. An overview of the general theory of functional equations can be found for example in [Kuczma et al. \(1990\)](#), [Kress et al. \(1989\)](#), [Polyanin and Manzhirov \(2008\)](#), or [Zemyan \(2012\)](#). There are interesting applications of this theory in fields as diverse as geometry, probability theory, financial management, or information theory.

We provide a separate treatment of the associated linear functional equation of higher order for the cases where market parameters are homogeneous and heterogeneous throughout a performance evaluation period in Sections 3 and 4 respectively. For the case of homogeneous market parameters, we adapt the techniques of [Kuczma et al. \(1990\)](#) to reduce the original higher-order equation to a system of linear equations of order one. This technique cannot be directly extended to the more complex case of heterogeneous market parameters. We approach this case by first deriving a relation between a sequence of single-period inverse investment problems and the multi-period counterpart. Based on this connection, we then establish an equivalence between the original functional equation of higher order and a system of equations of order one related to the construction of an auxiliary single-period forward process. In addition

to the explicit construction, we also show that discrete-time predictable forward performance processes are decreasing in the evaluation period. In continuous time, forward performance processes are not necessarily monotone in time. However, continuous-time forward performance processes that are time-monotone often allow for more explicit results, see, e.g., [Musielà and Zariphopoulou \(2009\)](#) and [Berrier et al. \(2009\)](#).

The second major contribution of this paper is an application of multi-period discrete-time predictable forward processes as a framework to model preferences for robo-advising. To the best of our knowledge, this is the first application of the forward theory to the asset allocation problem faced by a robo-advisor. Robo-advisors constitute a class of wealth management advisors that offer asset allocation recommendations and implementations based on algorithms automated by software (see, for example, [Beketov et al. \(2018\)](#), [Capponi et al. \(2021\)](#), [Cui et al. \(2019\)](#), [Alsabah et al. \(2021\)](#), or [Dai et al. \(2021b\)](#)). Forward processes have three important features making them expedient for robo-advising applications.

First, the construction of forward processes assures that optimal investment strategies are time-consistent. This is in stark contrast to the dynamic mean-variance objective. Whenever preferences are time-inconsistent, one has to decide on whether to work with pre-committed or equilibrium strategies, and there does not seem to be a canonic choice for robo-advising applications. [Capponi et al. \(2021\)](#) and [Dai et al. \(2021a\)](#) work with equilibrium strategies while [Cui et al. \(2019\)](#) introduce a mean-variance induced utility functions to avoid the issue altogether. However, it seems also plausible to work with pre-committed strategies and regard the robo-advisor as a pre-committment device. Working with forward processes avoids this discussion and leads to strategies that are globally optimal.

Second, forward processes accommodate dynamically changing investment horizons. While this feature is an advantage for portfolio selection in general, it is of particular relevance for robo-advising applications. Imagine a situation where the investment horizon of a client is reached, but the client does not withdraw her funds. How should the robo-advisor act in this situation if it aims to continue investing in the best interest of the client? Forward preferences

provide an elegant solution to this problem: Continue investing in a manner that is consistent with previous preferences and decisions by updating preferences according to the martingale optimality principle.

Third, forward preferences allow for a dynamic updating of the model for the financial market. In the context of robo-advising, this allows us to integrate changing beliefs of the client into the dynamic asset allocation process. Integrating personal beliefs about the distributional characteristics of the risky assets into the portfolio optimization process has a long tradition starting from the seminal work of [Black and Litterman \(1991, 1992\)](#).

In addition to these general advantages of forward performance processes, the specific class we investigate herein allow for the additional feature that trading times do not necessarily coincide with performance evaluation times. This is of practical relevance for robo-advising applications where trading typically occurs at a higher frequency than interaction with the client. We thus consider a client of a robo-advisor whose preferences are described by a discrete-time predictable performance process. On the one hand, the client has time-varying beliefs about the financial market and communicates these at infrequently occurring interactions with the robo-advisor. On the other hand, the robo-advisor manages the portfolio on behalf of the client period-by-period based on the assessment of the market the client communicated at the last interaction time. The client seeks to determine an optimal schedule for interacting with the robo-advisor that balances a tradeoff between accuracy about the current beliefs of the client about the market parameters and an interaction cost reflecting the time and effort required to interact with the robo-advisor. This problem is inspired by [Capponi et al. \(2021\)](#), who analyze a different tradeoff, namely between uncertainty about the risk preferences of the agent and behavioral biases in the preferences communicated by the agent. Different from their setting, we do not consider uncertainty over preferences and behavioral biases the client might have when communicating her preferences. Instead, we consider uncertainty about the client's beliefs about the financial market.

We study the problem of determining an optimal interaction schedule for an investor whose

initial preferences are described by a CRRA utility function under two alternative criteria: a robust approach over a set of possible evolutions of beliefs and an explicit updating rule where the probability of an upward movement of the stock is a maximum likelihood estimator. For the robust criterion, we characterize the optimal interaction schedule and find that it balances a tradeoff between interaction cost and expected loss in performance due to the inaccuracy about the market parameters. As one could intuitively expect, the optimal interaction schedule is increasing in the interaction cost and decreasing in a uniform increase of uncertainty about the market parameters. However, the effect of a non-uniform increase in the uncertainty is more intricate, and it can indeed happen that the optimal interaction schedule increases when uncertainty about the market parameters in the near future increases. This occurs because an increase in the uncertainty about market parameters in the near future harms performance after each interaction time. Interacting more frequently therefore does not necessarily lead to better performance. We also numerically investigate how the optimal interaction schedule depends on the risk-aversion of the client. Typically, a more risk-averse client is interacting more frequently with the robo-advisor than a less risk-averse client. However, when the interaction cost is large and either the expected return of the risky asset is close to the risk-free return or the risk-aversion is already large, then an increase in risk-aversion can lead to an increase of the optimal interaction schedule. In this case, the investment in the risky asset is very small, and the updating of the probability for a positive outcome does not lead to a significant change in optimal investment strategies. Numerical studies under the criteria where the probability for an upward movement of the stock is updated according to a maximum likelihood criterion largely conform with the above findings.

The remainder of this paper is organized as follows. In Section 2, we introduce the model for the financial market and review the definition and preliminary results for discrete-time predictable performance processes. We provide sufficient conditions for existence and uniqueness and an explicit construction of the discrete-time predictable forward process for homogeneous market parameters in Section 3 and for heterogeneous market parameters in Section 4. In

Section 5, we discuss discrete-time predictable forward processes as a potential framework to model preferences for robo-advising applications. Section 6 concludes the paper.

2 Discrete-time predictable forward performance processes: Model and definition

In this section, we introduce the notion of discrete-time predictable forward performance process with evaluation period larger than one in the binomial tree model which was originally presented in Cox et al. (1979) for option pricing. Discrete-time predictable forward performance processes were introduced in Angoshtari et al. (2020) for general models of the financial market. However, their analysis is limited to the single-period binomial model where trading dates and performance evaluation dates coincide. The complete semimartingale model in Strub and Zhou (2021) is more general than the setup of this paper, but they do not provide conditions for existence and do not explicitly construct discrete-time predictable forward processes as we will herein.

The investor starts at time $t_0 = 0$ with preferences over wealth represented by a utility function U_0 . We herein assume that any utility function $U : \mathbb{R}^+ \rightarrow \mathbb{R}$ is twice continuously differentiable, strictly increasing, strictly concave and satisfies the Inada conditions. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where \mathbb{P} denotes the real (historical) probability measure on (Ω, \mathcal{F}) . Throughout the paper, \mathbb{N} denotes the set of positive integers and \mathbb{N}_0 is the set of nonnegative integers. The probability space is endowed with a filtration $\mathbb{F} = (\mathcal{F}_n)_{n \in \mathbb{N}_0}$. We suppose that the investor trades between a risk-free bond and a single stock at discrete times t_n , $n \in \mathbb{N}_0$, and evaluates her portfolio at performance evaluation times $(\tau_k)_{k \in \mathbb{N}_0}$, which is a stochastic process taking values in $\{t_n, n \in \mathbb{N}_0\}$ such that $\tau_0 = 0$, $\tau_{k+1} > \tau_k$ and τ_k is $\mathcal{F}_{\tau_{k-1}}$ measurable. This measurability requirement implies that the length of each evaluation period is known at the beginning of the respective period. Unless mentioned otherwise, we consider a fixed *evaluation period length* $m, m \in \mathbb{N}$, i.e., $\tau_k = t_{km}$, $k \in \mathbb{N}_0$. This assumption is not crucial and made

primarily for safe notation. Additionally, for a simplified notation, we will from now on assume that $t_k = k$.

When $m = 1$, trading times and performance evaluation times coincide and the model reduces to the one extensively studied in [Angoshtari et al. \(2020\)](#). However, in general, the evaluation period length is strictly larger than one and trading thus occurs at a higher frequency than performance evaluation. This separation between trading times and performance evaluation times is a key feature of our model and will be at the heart of our analysis and application. We remark that we make an implicit assumption that trading is more frequent than performance evaluation and the investor trades simultaneously whenever evaluating her performance. This is natural. Performance evaluation without concurrent trading would not be observable.

Let R_n denote the total return of the stock over period $[n - 1, n)$. The risk-free bond does not offer any interest. The return process $R = (R_n)_{n \in \mathbb{N}}$ is adapted and R_n takes on two possible values, u_n and d_n , with probabilities $p_n := \mathbb{P}[R_n = u_n | \mathcal{F}_{n-1}]$ and $1 - p_n = \mathbb{P}[R_n = d_n | \mathcal{F}_{n-1}]$ respectively. The market parameters $(d_n)_{n \in \mathbb{N}}$, $(u_n)_{n \in \mathbb{N}}$, and $(p_n)_{n \in \mathbb{N}}$ are allowed to be random processes. We assume that $u_{(k-1)m+j}$, $d_{(k-1)m+j}$, and $p_{(k-1)m+j}$ are $\mathcal{F}_{(k-1)m}$ -measurable for every $k \in \mathbb{N}$ and $j \in \{1, 2, \dots, m\}$. This assures that all market parameters are known at the beginning of each *(performance) evaluation period* $[(k - 1)m, km]$. The measurability assumptions on the market parameters are critical for the results presented herein and relaxing them would present a significant challenge. In order to assure absence of arbitrage, we further assume that $0 < d_n < 1 < u_n$ and that $0 < p_n < 1$. Under these assumptions, the risk-neutral probability measure \mathbb{Q} with $q_n = \mathbb{Q}[R_n = u_n | \mathcal{F}_{n-1}] = 1 - \mathbb{Q}[R_n = d_n | \mathcal{F}_{n-1}]$ given by $q_n = \frac{1-d_n}{u_n-d_n} \in (0, 1)$, $n \in \mathbb{N}$, is equivalent to \mathbb{P} and q_n is the unique risk-neutral probability for an upward movement of the stock in the period $[n - 1, n]$.

Trading strategies are described by means of predictable processes $\pi = (\pi_n)_{n \in \mathbb{N}}$, where π_n denotes the dollar amount invested in the risky asset over trading period $[n - 1, n)$. A portfolio is constructed by following the trading strategy on the stock while investing all the remaining

wealth in the risk-free bond. Given an initial wealth $x > 0$ and self-financing trading strategy π , the wealth process $X^\pi = (X_n^\pi)_{n \in \mathbb{N}}$ evolves according to $X_n^\pi = x + \sum_{i=1}^n \pi_i(R_i - 1)$. A trading strategy π as well as the associated wealth process X^π are called *admissible* if X^π is nonnegative. We denote by $\mathcal{A}(n, x)$ the set of admissible trading strategies $(\pi_k)_{k \geq n}$ and by $\mathcal{X}(n, x)$ the associated wealth processes $(X_k^\pi)_{k \geq n}$ starting from $X_n^\pi = x$, $n \in \mathbb{N}$, and abbreviate $\mathcal{A}(0, x)$, $\mathcal{X}(0, x)$ by $\mathcal{A}(x)$, $\mathcal{X}(x)$. We often drop the explicit dependence of a wealth process on the trading strategy and write $X \in \mathcal{X}(n, x)$.

We next present the definition of discrete-time predictable forward performance processes with evaluation period length m .

Definition 1. *A family of random functions $\{U_{km} : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R} | k \in \mathbb{N}_0\}$ is called a discrete-time predictable forward performance process with evaluation period length $m \in \mathbb{N}$ (a m -forward process in short) if the following conditions hold:*

- (i) $U_0(x, \cdot)$ is constant and $U_{km}(x, \cdot)$ is $\mathcal{F}_{(k-1)m}$ -measurable for each $x \in \mathbb{R}^+$ and $k \in \mathbb{N}$.
- (ii) $U_{km}(\cdot, \omega)$ is a utility function for almost all $\omega \in \Omega$ and all $k \in \mathbb{N}_0$.
- (iii) For any initial wealth $x > 0$ and admissible wealth process $X \in \mathcal{X}(x)$,

$$U_{(k-1)m}(X_{(k-1)m}) \geq \mathbb{E}[U_{km}(X_{km}) | \mathcal{F}_{(k-1)m}], \quad k \in \mathbb{N}.$$

- (iv) For any initial wealth $x > 0$, there exists an admissible wealth process $X^* \in \mathcal{X}(x)$ such that

$$U_{(k-1)m}(X_{(k-1)m}^*) = \mathbb{E}[U_{km}(X_{km}^*) | \mathcal{F}_{(k-1)m}], \quad k \in \mathbb{N}.$$

Definition 1 is analogous to its single-period counterpart, but we are now interested in the case where trading occurs more often than performance evaluation. See [Angoshtari et al. \(2020\)](#) for a detailed discussion of the definition and a theoretical framework of discrete-time

predictable forward performance processes. Considering discrete-time predictable forward performance process with evaluation period larger than one is more general mathematically and also relevant for applications. It is increasingly the case that trading is automated and executed by machines at a higher frequency than monitoring and analyzing of the investment portfolio by a human agent. Modelling a framework where trading occurs at a higher frequency than performance evaluation and preference updating is thus important for investment practice.

Property (i) requires that preferences applying at the end of an evaluation period are known at the beginning of that period. This reflects the *predictability* of discrete-time predictable forward processes adapted to multi-period evaluation of the performance. Properties (iii) and (iv) demand that an m -forward process evolves under the guidance of Martingale Optimality Principle and ensure time-consistency of optimal strategies. In addition, properties (iii) and (iv) imply that

$$U_{(k-1)m}(X_{(k-1)m}^*) = \operatorname{ess\,sup}_{X_{km} \in \mathcal{X}((k-1)m, X_{(k-1)m}^*)} \mathbb{E} \left[U_{km}(X_{km}) \middle| \mathcal{F}_{(k-1)m} \right]. \quad (1)$$

Iteratively solving (1) leads to the construction of the m -forward process, see [Angoshtari et al. \(2020\)](#) for a detailed exposition. The crucial step is to solve the following *inverse investment problem*: Given an initial utility function U_0 , we seek for a forward utility function U_m such that for any $x > 0$,

$$U_0(x) = \operatorname{ess\,sup}_{X_m \in \mathcal{X}(x)} \mathbb{E}[U_m(X_m)] = \sup_{\pi \in \mathcal{A}(x)} \mathbb{E} \left[U_m \left(x + \sum_{i=1}^m \pi_i (R_i - 1) \right) \right]. \quad (2)$$

One can then construct U_{2m}, U_{3m}, \dots by repeatedly solving problem of the form (2) conditionally on updated information available at next evaluation point and arguing that this solution satisfies the required measurability conditions. We emphasize that obtaining a solution that is measurable as a function of the market parameters is necessary for the construction of a predictable forward process, cf. [Strub and Zhou \(2021, Remark 2.2\)](#) for details.

Remark 1. *When deriving the solution to this inverse investment problem (2), we argue that*

the constructed forward utility function depends in a measurable way on all market parameters at the previous evaluation time, and that this will allow us to obtain a predictable process. Therefore, the dynamic version of the sequence of random problems (1) can be reduced to the deterministic version (2).

In analogy to the terminology in Strub and Zhou (2021), we will refer to an initial utility function U_0 and a utility function U_m solving (2) as an m -forward pair (U_0, U_m) . Note that our assumptions imply that the model input is known at the beginning for the evaluation period as a deterministic triplet $((p_i)_{i=1,\dots,m}, (u_i)_{i=1,\dots,m}, (d_i)_{i=1,\dots,m})$. With slight abuse of notation, we denote the unique equivalent probability measure on this truncated model only for single evaluation period again by \mathbb{Q} .

A key result for the theory of discrete-time predictable forward processes is the equivalence between the inverse investment problem (2) and a generalized integral equation for the inverse marginal or the conjugate corresponding to the involved forward pair. This was shown for the binomial market in Angoshtari et al. (2020) and generalized to complete semimartingale models in Strub and Zhou (2021). To state this result, we recall the definition of an inverse marginal function. An *inverse marginal function* $I(y) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuously differentiable, strictly decreasing and satisfies $\lim_{y \rightarrow +\infty} I(y) = 0$ and $\lim_{y \rightarrow 0^+} I(y) = \infty$. For a given utility function $U(x), x \in \mathbb{R}^+$, $I(y) = (U')^{-1}(y)$ is the inverse marginal function corresponding to $U(x)$. We denote the set of utility functions by \mathcal{U} , the set of inverse marginal functions by \mathcal{I} . According to Theorem 2.4 in Strub and Zhou (2021), see also Theorems 5.1 and 5.2 in Angoshtari et al. (2020) for an earlier version in the single-period binomial setting, solving the inverse investment problem (2) in the space \mathcal{U} of utility functions is equivalent to finding a solution to

$$I_0(\hat{y}) = \mathbb{E}_{\mathbb{Q}} \left[I_m \left(\hat{y} \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right], \quad \hat{y} > 0, \quad (3)$$

in the space \mathcal{I} of inverse marginal functions in the following sense: If (U_0, U_m) is an m -forward pair solving (2), then the associated inverse marginal functions (I_0, I_m) solve (3). Vice versa, if

(I_0, I_m) is a pair of inverse marginal functions satisfying (3), then the associated utility functions satisfy (2) up to a constant, i.e., there is a constant $c \in \mathbb{R}$, which can be expressed explicitly in terms of U_0 , I_m , and the market parameters, such that $\tilde{U}_m(x) := U_m(x) + c$ satisfies (2). Because it is often the case that finding a solution to the generalized integral equation (3) is considerably easier than solving the inverse investment problem (2), the generalized integral equation (3) plays an important role in the theory of discrete-time predictable forward processes. Our main technical contribution is to provide a solution to (3) for the binomial market when trading times do not coincide with performance evaluation times, and thus (3) reduces to a linear functional equation as in Angoshtari et al. (2020) but of higher order. Solving (3), together with a thorough analysis of the result, will be the content of the following Sections 3 and 4 for the case of time-homogeneous and time-heterogeneous market parameters respectively.

3 The case of time-homogeneous market parameters

We first develop an iterative method to solve the linear functional equation (3) associated with the inverse investment problem (2) for the setting of *time-homogeneous market parameters*. Specifically, we suppose for this section that the deterministic triplet $((p_i)_{i=1,\dots,m}, (u_i)_{i=1,\dots,m}, (d_i)_{i=1,\dots,m})$ of the multi-period binomial tree remains constant for different trading periods within one evaluation period, $p_i = p$, $u_i = u$, and $d_i = d$ for $i = 1, \dots, m$. Thus, while market parameters are still updated at the beginning of each evaluation period, we assume for this section that they are constant throughout each evaluation period. This slight loss of generality allows us to derive more explicit and interpretable results.

Following Angoshtari et al. (2020), we set $a = \frac{1-p}{p} \frac{q}{1-q}$, $b = \frac{1-q}{q}$, $c = \frac{1-p}{1-q}$. When trading occurs more frequently than performance evaluation, the price process of the risky asset in a given evaluation period corresponds to an m -period binomial tree with homogeneous coefficients: There are $m + 1$ possible outcomes which, when ordered from the lowest price level to the highest, occur with the probabilities $\binom{m}{i} p^i (1-p)^{m-i}$, $i = 0, 1, \dots, m$, where the transition probability is denoted by p and $\binom{m}{i} = \frac{m!}{i!(m-i)!}$ are binomial coefficients. Therefore, (3) can be

written as

$$I_0(\hat{y}) = \sum_{i=0}^m \binom{m}{i} q^i (1-q)^{m-i} I_m \left(\hat{y} \frac{q^i (1-q)^{m-i}}{p^i (1-p)^{m-i}} \right), \quad \hat{y} > 0. \quad (4)$$

The main technical contribution of this section will be a characterization of solutions to the linear functional equation (4) in the class of inverse marginal functions including conditions for uniqueness.

For a given initial utility function $U_0 \in \mathcal{U}$ and associated inverse marginal function $I_0 \in \mathcal{I}$ we define the following auxiliary functions,

$$\Phi_0^0(y) = I_0(ac^m y) - bI_0(c^m y) \quad \text{and} \quad \Psi_0^0(y) = y^{-\log_a b} I_0(c^m y), \quad y > 0, \quad (5)$$

and

$$\begin{aligned} \Phi_i^j(y) &= \frac{(1+b)^m}{b^j} \left(\sum_{n_1=0, \dots, n_i=0}^{\infty} (-1)^{p(n_1, \dots, n_i)} b^{q_{j;(n_1, \dots, n_i)}} I_0 \left(a^{r_{j;(n_1, \dots, n_i)}+1} c^m y \right) \right. \\ &\quad \left. - b \sum_{n_1=0, \dots, n_i=0}^{\infty} (-1)^{p(n_1, \dots, n_i)} b^{q_{j;(n_1, \dots, n_i)}} I_0 \left(a^{r_{j;(n_1, \dots, n_i)}} c^m y \right) \right), \\ \Psi_i^j(y) &= y^{-\log_a b} \frac{(1+b)^m}{b^j} \sum_{n_1=0, \dots, n_i=0}^{\infty} (-1)^{p(n_1, \dots, n_i)} b^{q_{j;(n_1, \dots, n_i)}} I_0 \left(a^{r_{j;(n_1, \dots, n_i)}} c^m y \right), \end{aligned} \quad (6)$$

for $y > 0$, $i = 1, \dots, m-1$ and $j = 0, 1, \dots, i$, where the exponents are defined as $p(n_1, \dots, n_i) = \sum_{k=1}^i n_k$, $q_{j;(n_1, \dots, n_i)} = -\sum_{k=1}^j n_k + \sum_{k=j+1}^i n_k$, and $r_{j;(n_1, \dots, n_i)} = \sum_{k=1}^j n_k - \sum_{k=j+1}^i (n_k + 1)$.

For a given pair of functions (Φ, Ψ) , we say that the pair satisfies condition (C1) if

$$\Phi'(y) > 0 \text{ and either } a > 1 \text{ and } \lim_{y \rightarrow \infty} \Psi(y) = 0 \text{ or } a < 1 \text{ and } \lim_{y \rightarrow 0^+} \Psi(y) = 0.$$

We say that the pair of functions (Φ, Ψ) satisfies condition (C2) if

$$\Phi'(y) < 0 \text{ and either } a > 1 \text{ and } \lim_{y \rightarrow 0^+} \Psi(y) = 0 \text{ or } a < 1 \text{ and } \lim_{y \rightarrow \infty} \Psi(y) = 0.$$

Next, we iteratively define the sequence $(A_i)_{i=0,\dots,m}$ starting with $A_0 = 0$ by setting

$$A_{i+1} = \begin{cases} A_i + 1 & \text{if } (\Phi_i^{A_i}, \Psi_i^{A_i}) \text{ satisfies (C1),} \\ A_i & \text{if } (\Phi_i^{A_i}, \Psi_i^{A_i}) \text{ satisfies (C2),} \end{cases}$$

for $i = 0, \dots, m-1$.

The sequence $(A_i)_{i=0,\dots,m}$ is typically well defined for CRRA utility functions $U_0(x) = \log x$, $x > 0$ and $U_0(x) = (1 - \frac{1}{\theta})^{-1} x^{1-\frac{1}{\theta}}$, $x > 0$, where $1 \neq \theta > 0$. Exceptions are the cases where $p = \frac{1}{2}$ or $\theta = -\log_a b$. In these cases, (Φ_i^j, Ψ_i^j) satisfy neither (C1) nor (C2) for any i, j , but one can still provide a natural candidate for the forward process within the family of power and log utilities and show that this is indeed a forward process. However, uniqueness generally does not hold in this case, ([Angoshtari et al., 2020](#), Example 6.1). Therefore, we emphasize that the condition that $(A_i)_{i=0,\dots,m}$ exists is sufficient, but not necessary for the existence and uniqueness of the forward process. How to solve the corresponding functional equation and construct the forward performance process when $(A_i)_{i=0,\dots,m}$ does not exist remains an open problem for future research.

We now state the main result of this section which yields a *construction method* for an m -forward pair and presents an explicit relationship between the associated inverse marginal functions, I_0 and I_m . This theorem is the multi-period analogue to the single-period result in ([Angoshtari et al., 2020](#), Theorem 7.1).

Theorem 1. *Let $U_0 \in \mathcal{U}$ be a utility function with associated inverse marginal function I_0 and suppose that $(A_i)_{i=0,\dots,m}$ exists. Define $I_m : (0, \infty) \rightarrow (0, \infty)$ by*

$$I_m(y) := \frac{(1+b)^m}{b^{A_m}} \sum_{n_1=0,\dots,n_m=0}^{\infty} (-1)^{p(n_1,\dots,n_m)} b^{q_{A_m:(n_1,\dots,n_m)}} I_0(a^{r_{A_m:(n_1,\dots,n_m)}} c^m y), \quad y > 0, \quad (7)$$

and $U_m : (0, \infty) \rightarrow (0, \infty)$ by

$$U_m(x) := U_0(1) + \mathbb{E}_{\mathbb{P}} \left[\int_{I_m(\frac{dQ}{dP} U_0'(1))}^x I_m^{-1}(t) dt \right], \quad x > 0.$$

Then U_m is the unique utility function solving the inverse investment problem (2) and I_m is the unique inverse marginal function solving the linear functional equation (4). Moreover, the optimal wealth solving (2) is given by

$$X_m^*(x) = I_m \left(U_0'(x) \frac{d\mathbb{Q}}{d\mathbb{P}} \right).$$

From the explicit construction of an m -forward pair in Theorem 1, we obtain the following corollary showing that the forward utility U_m depends in a measurable manner on the parameters of the financial market. This measurable dependence is crucial because it allows us to extend all results derived for an m -forward pair back to the level of a discrete-time predictable forward performance process with evaluation period length m .

Corollary 1. *Let $U_0 \in \mathcal{U}$ be a utility function and let*

$$\mathcal{M} := \{(p, u, d) \in \mathbb{R}^3 \mid 0 < p < 1, 0 < d < 1 < u, (A_i)_{i=0,\dots,m} \text{ exists}\}$$

be the set of market parameters under which $(A_i)_{i=0,\dots,m}$ exists. The mapping $\mathcal{M} \rightarrow \mathbb{R}$ defined by $(p, u, d) \mapsto U_m(x)$, where $U_m(x)$ is defined as in Theorem 1, is Borel-measurable for any $x > 0$.

Having established an explicit construction of an m -forward pair in Theorem 1, we next present the comparison between the discrete-time predictable forward performance process with evaluation period length m and the single-period discrete-time forward process after m -periods of updating when the market parameters are homogeneous. We denote the latter process by $\tilde{U} = (\tilde{U}_k)_{k \in \mathbb{N}_0}$ and are interested in comparing \tilde{U}_m with U_m . Given an initial performance criterion U_0 and the homogeneous market parameters (p, u, d) , the process $\{\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_m\}$ is constructed according to the general scheme outlined in Section 7 of Angoshtari et al. (2020).

Proposition 1. *Assume that the model input parameters (p, u, d) are deterministic and remain constant for m consecutive periods. If $(A_i)_{i=0,\dots,m}$ exists, then the single-period forward process*

\tilde{U}_i exists for $i = 1, \dots, m$, and satisfies $U_m(x) = \tilde{U}_m(x)$ for all $x > 0$.

Furthermore, the optimal wealth processes corresponding to the m -forward and period-by-period forward coincide as well. Indeed, denoting the optimal wealth process corresponding to \tilde{U} by \tilde{X} , Theorem 2.0 in [Kramkov and Schachermayer \(1999\)](#) yields that

$$\begin{aligned}\tilde{X}_m^*(x) &= \tilde{I}_m \left(\rho_m \tilde{U}'_{m-1} \left(\tilde{X}_{m-1}^*(x) \right) \right) \\ &= \tilde{I}_m \left(\rho_m \tilde{U}'_{m-1} \left(\tilde{I}_{m-1} \left(\rho_{m-1} \tilde{U}'_{m-2} \left(\tilde{X}_{m-2}^*(x) \right) \right) \right) \right) \\ &= \tilde{I}_m \left(\rho_m \rho_{m-1} \tilde{U}'_{m-2} \left(\tilde{X}_{m-2}^*(x) \right) \right) \\ &= \tilde{I}_m \left(\rho_m \rho_{m-1} \times \dots \times \rho_1 U'_0(x) \right) \\ &= I_m \left(\frac{dQ}{dP} U'_0(x) \right),\end{aligned}$$

where for $i = 1, 2, \dots, m$, $\rho_i = \frac{q}{p} \mathbb{1}_{\{R_i=u\}} + \frac{1-q}{1-p} \mathbb{1}_{\{R_i=d\}}$ by assumption of homogeneous market parameters.

Proposition 1 allows us to establish the following result showing that the m -forward utility U_m is monotone in the duration of the evaluation period.

Proposition 2. *Let U_0 be an initial utility function and suppose that $(A_i)_{i=1,\dots,m}$ exists for any $m \in \mathbb{N}$. Then the m -forward $U_m(x)$ is non-increasing for any fixed $x \in \mathbb{R}^+$ in the length of evaluation period $m \in \mathbb{N}$, and strictly decreasing if the expected excess return of the risky asset is non-zero.*

This result is consistent with investors' intuition that the utility which can be derived from a fixed monetary amount decreases over time. An example with power utility function in which the forward performance can be computed explicitly is given below as an illustration conforming with Proposition 2.

Example 1. *Let $U_0(x) = (1 - \frac{1}{\theta})^{-1} x^{1-\frac{1}{\theta}}$, $x > 0$, and assume that $1 \neq \theta > 0, \theta \neq -\log_a b$. By*

applying Theorem 1,

$$\begin{aligned} I_m(x) &= \left(\frac{1+b}{c^\theta(a^{-\theta}+b)} \right)^m y^{-\theta}, \\ U_m(x) &= \left(\frac{1+b}{c^\theta(a^{-\theta}+b)} \right)^{m/\theta} U_0(x), \\ U_{m+1}(x) &= \left(\frac{1+b}{c^\theta(a^{-\theta}+b)} \right)^{(m+1)/\theta} U_0(x) = \left(\frac{1+b}{c^\theta(a^{-\theta}+b)} \right)^{1/\theta} U_m(x) \end{aligned}$$

We now discuss the value of $\left(\frac{1+b}{c^\theta(a^{-\theta}+b)} \right)^{1/\theta}$ and show through direct computation that $U_{m+1}(x) \leq U_m(x)$. Recalling that $a = \frac{1-p}{p} \frac{q}{1-q}$, $b = \frac{1-q}{q}$, $c = \frac{1-p}{1-q}$ we have $\frac{1+b}{c^\theta(a^{-\theta}+b)} = \frac{1}{q(\frac{p}{q})^\theta + (1-q)(\frac{1-p}{1-q})^\theta}$. Define $f(t) = q(\frac{p}{q})^t + (1-q)(\frac{1-p}{1-q})^t - 1$. Clearly, $f(t)$ is strictly convex, $f(0) = 0$, $t_1 = 0$ is one root of $f(t) = 0$. According to Jensen's inequality, we have $f'(0) = q \log(\frac{p}{q}) + (1-q) \log(\frac{1-p}{1-q}) < \log(q\frac{p}{q} + (1-q)\frac{1-p}{1-q}) = 0$. Hence, there is another strictly positive root $t_2 = 1$ of $f(t) = 0$. Therefore, $f(\theta) < 0$, if $0 < \theta < 1$ and $f(\theta) > 0$, if $\theta > 1$. When $0 < \theta < 1$, $f(\theta) < 0$, $\left(\frac{1+b}{c^\theta(a^{-\theta}+b)} \right)^{1/\theta} > 1$ and $U_m(x) < 0$, then $U_{m+1}(x) < U_m(x)$. Analogously, when $\theta > 1$, $f(\theta) > 0$, $\left(\frac{1+b}{c^\theta(a^{-\theta}+b)} \right)^{1/\theta} < 1$ and $U_m(x) > 0$, then we still have $U_{m+1}(x) < U_m(x)$. This example with power utility functions illustrates Proposition 2 showing that the forward performance process is strictly decreasing as the frequency of evaluation decreases, or equivalently, m increases.

We close this example by investigating which combinations of the market parameters lead to the largest decreases of the forward performance process. We first consider the case when $\theta > 1$, and thus $U_m(x) > 0$, and define the auxiliary function $g(p) = \frac{1}{q(\frac{p}{q})^\theta + (1-q)(\frac{1-p}{1-q})^\theta}$. Straightforward computations show that $g(p)$ will first increase, attain its maximum at $p = q$ and then decrease, and that $g(0) = (1-q)^{\theta-1}$, $g(1) = q^{\theta-1}$. Therefore, we conclude that when $q < \frac{1}{2}$, p tending to 1 yields the largest decrease, while when $q > \frac{1}{2}$, p tending to 0 yields the largest decrease. An analogous analysis holds for $\theta < 1$. We thus conclude that the combinations of (p, q) approaching $(1, 0)$ or $(0, 1)$ maximise the decrease of the forward utility $U_m(x)$, $m \in \mathbb{N}$. Generally speaking, when the risk-neutral probabilities deviate to the highest level from the historical real-world measure, the forward performance process decreases fastest.

4 The case of time-heterogeneous market parameters

In this section, we generalize the previous results to allow for an agent with heterogeneous beliefs on future price movements across the trading periods constituting a given evaluation period. Specifically, we consider the case where the deterministic triplet $((p_i)_{i=1,\dots,m}, (u_i)_{i=1,\dots,m}, (d_i)_{i=1,\dots,m})$ characterizing the multi-period binomial tree is heterogeneous in time. We accordingly set $a_i = \frac{1-p_i}{p_i} \frac{q_i}{1-q_i}$, $b_i = \frac{1-q_i}{q_i}$, $c_i = \frac{1-p_i}{1-q_i}$ for $i = 1, 2, \dots, m$. Observe that there are 2^m possible outcomes for the m -period binomial tree with heterogeneous market parameters. When ordered from the lowest price level to the highest, they occur with probabilities $\prod_{i=1}^m p_i^{\gamma_{j,i}} (1-p_i)^{1-\gamma_{j,i}}$, $j = 0, 1, \dots, 2^m - 1$, where $\gamma_{j,i}$ is defined as the i 'th digit of the binary representation of j , i.e., $(j)_{10} = (\gamma_{j,m} \dots \gamma_{j,2} \gamma_{j,1})_2$, where zeros are filled in the front of the binary representation if it contains less than m digits. In the current setting, the generalized integral equation (3) can thus be written as the linear functional equation

$$I_0(\hat{y}) = \sum_{j=0}^{2^m-1} \prod_{i=1}^m q_i^{\gamma_{j,i}} (1-q_i)^{1-\gamma_{j,i}} I_m \left(\frac{\prod_{i=1}^m q_i^{\gamma_{j,i}} (1-q_i)^{1-\gamma_{j,i}}}{\prod_{i=1}^m p_i^{\gamma_{j,i}} (1-p_i)^{1-\gamma_{j,i}}} \hat{y} \right). \quad (8)$$

Analyzing (8) is more challenging than the homogeneous analogue (4) because the argument of I_m can in general not be transformed to an iterative form. However, we are still able to characterize solutions to (8) within the class of inverse marginal functions and provide conditions for the uniqueness of such a solution. This will be the main technical contribution of this section.

For a given initial utility function $U_0 \in \mathcal{U}$ and associated inverse marginal function $I_0 \in \mathcal{I}$ we define the following auxiliary functions,

$$\Phi_0(y) = I_0(a_1 c_1 y) - b_1 I_0(c_1 y) \quad \text{and} \quad \Psi_0(y) = y^{-\log_{a_1} b_1} I_0(c_1 y), \quad y > 0, \quad (9)$$

$$\begin{aligned}
 \Phi_i^{(v_1, \dots, v_i)} &= \frac{\prod_{l=1}^i (1 + b_l)}{\prod_{j=1}^m b_j^{v_j}} \left(\sum_{n_1=0, \dots, n_i=0}^{\infty} (-1)^{p(n_1, \dots, n_i)} Q_{(v_1, \dots, v_i); (n_1, \dots, n_i)} I_0 \left(R_{(v_1, \dots, v_i); (n_1, \dots, n_i)} a_{i+1} y \right) \right. \\
 &\quad \left. - b \sum_{n_1=0, \dots, n_i=0}^{\infty} (-1)^{p_0(i)} Q_{(v_1, \dots, v_i); (n_1, \dots, n_i)} I_0 \left(R_{(v_1, \dots, v_i); (n_1, \dots, n_i)} y \right) \right), \\
 \Psi_i^{(v_1, \dots, v_i)} &= y^{-(\log_{a_{i+1}} b_{i+1})} \frac{\prod_{l=1}^i (1 + b_l)}{\prod_{j=1}^m b_j^{v_j}} \\
 &\quad \times \sum_{n_1=0, \dots, n_i=0}^{\infty} (-1)^{p(n_1, \dots, n_i)} Q_{(v_1, \dots, v_i); (n_1, \dots, n_i)} I_0 \left(R_{(v_1, \dots, v_i); (n_1, \dots, n_i)} y \right),
 \end{aligned} \tag{10}$$

for $y > 0$, $i = 1, \dots, m-1$, and $(v_1, \dots, v_i) \in \{0, 1\}^i$, where $Q_{(v_1, \dots, v_i); (n_1, \dots, n_i)} = \prod_{k=1}^i b_k^{n_k(1-2v_k)}$, $R_{(v_1, \dots, v_i); (n_1, \dots, n_i)} = \prod_{s=1}^i a_s^{n_s(2v_s-1)+(v_s-1)} \prod_{u=1}^{i+1} c_u$ and $p(n_1, \dots, n_i) = \sum_{k=1}^i n_k$. We next iteratively define the sequence $\{(\alpha_1, \dots, \alpha_i)\}_{i=1, \dots, m}$, which will play a similar role as $(A_i)_{i=1, \dots, m}$ in the previous Section 3. We start by setting $(\alpha_1) = (1)$ if (Φ_0, Ψ_0) satisfies (C1), or $(\alpha_1) = (0)$ if (Φ_0, Ψ_0) satisfies (C2), and then iteratively define

$$(\alpha_1, \dots, \alpha_{i+1}) = \begin{cases} (\alpha_1, \dots, \alpha_i, 1) & \text{if } \left(\Phi_i^{(\alpha_1, \dots, \alpha_i)}, \Psi_i^{(\alpha_1, \dots, \alpha_i)} \right) \text{ satisfies (C1),} \\ (\alpha_1, \dots, \alpha_i, 0) & \text{if } \left(\Phi_i^{(\alpha_1, \dots, \alpha_i)}, \Psi_i^{(\alpha_1, \dots, \alpha_i)} \right) \text{ satisfies (C2),} \end{cases}$$

for $i = 1, \dots, m-1$.

The following theorem constitutes an analogue of Theorem 1 and constitutes the main result of this section. It provides an explicit expression for I_m in terms of I_0 with their corresponding utility functions being an m -forward pair when market inputs are heterogeneous throughout each evaluation period, and this expression in turn leads to a construction method for the m -forward pair.

Theorem 2. *Let $U_0 \in \mathcal{U}$ be a utility function with associated inverse marginal function I_0 and*

suppose that $\{(\alpha_1, \dots, \alpha_i)\}_{i=1, \dots, m}$ exists. Define I_m by

$$I_m(y) = \frac{\prod_{i=1}^m (1 + b_i)}{\prod_{j=1}^m b_j^{\alpha_j}} \sum_{n_1=0, \dots, n_m=0}^{\infty} (-1)^{p(n_1, \dots, n_m)} \prod_{k=1}^m b_k^{n_k(1-2\alpha_k)} I_0 \left(\prod_{s=1}^m d_s^{n_s(2\alpha_s-1)+(\alpha_s-1)} \prod_{u=1}^m c_u y \right), \quad (11)$$

and

$$U_m(x) := U_0(1) + \mathbb{E}_{\mathbb{P}} \left[\int_{I_m(\frac{d\mathbb{Q}}{d\mathbb{P}} U'_0(1))}^x I_m^{-1}(t) dt \right], \quad x > 0.$$

Then U_m is the unique utility function solving (2) and I_m is the unique inverse marginal function solving the generalized integral equation (8). Moreover, the optimal wealth solving (2) is given by

$$X_m^*(x) = I_m \left(U'_0(x) \frac{d\mathbb{Q}}{d\mathbb{P}} \right).$$

Along the same lines as in the homogeneous case, the explicit construction of an m -forward pair in Theorem 2 leads to the following corollary showing that U_m is measurable in the market parameters. We stress again the importance of this result because it immediately allows us to extend the analysis herein on m -forward pairs to the level of discrete-time predictable forward performance processes with evaluation period length m .

Corollary 2. *Let $U_0 \in \mathcal{U}$ be a utility function and let*

$$\mathcal{M} := \{(p, u, d) \in \mathbb{R}^{m \times 3} \mid 0 < p_i < 1, 0 < d_i < 1 < u_i, \{(\alpha_1, \dots, \alpha_i)\}_{i=1, \dots, m} \text{ exists}\}$$

be the set of market parameters under which $\{(\alpha_1, \dots, \alpha_i)\}_{i=1, \dots, m}$ exists and p, u , and d denote the $m \times 1$ vectors $(p_i)_{i=1}^m$, $(u_i)_{i=1}^m$, and $(d_i)_{i=1}^m$ respectively. The mapping $\mathcal{M} \rightarrow \mathbb{R}$ defined by $(p, u, d) \mapsto U_m(x)$, where $U_m(x)$ is defined as in Theorem 2, is Borel-measurable for any $x > 0$.

The analysis of this section showed that the heterogeneous case is not essentially different from the homogeneous setting. Allowing for heterogeneous market parameters gives more flexibility to model the financial market, but this comes at the expense of more complicated notation and formulas.

5 Robo-advising applications

Personalized robo-advisors provide automatized advice on asset allocation and investment strategies. They provide wealth management services for large number of clients and at lower cost than traditional financial advisors. Robo-advising companies constitute a rapidly growing part of the financial industry and are a prime example of FinTech, the application of technology to improve financial services. In this section, we propose and discuss discrete-time predictable forward performance processes as a potential framework for guiding asset allocation decisions of robo-advisors.

5.1 Preference modelling for robo-advising applications

Although robo-advising has rapidly grown in popularity over the last decade and now constitutes an important segment of modern investment industry, there is surprisingly little existing research on preference modeling for robo-advising applications and on the quantitative modelling of asset allocation decisions within those systems. [Capponi et al. \(2021\)](#) and [Cui et al. \(2019\)](#) were the first papers discussing the portfolio optimization part of robo-advisors quantitatively. While [Capponi et al. \(2021\)](#) proposed an adaptive mean-variance control model with updating of the risk aversion for deriving optimal allocation policies, [Cui et al. \(2019\)](#) considered the framework of mean-variance induced utility functions and argued that this approach has several desirable features from the perspective of robo-advisors. A further important study is [Dai et al. \(2021b\)](#) who consider the mean-variance objective for log returns introduced in [Dai et al. \(2021a\)](#), and provide an explicit formula for eliciting preferences in this setting. A

comparison of the key features of asset allocation models for robo-advising is given in Table 1.

Table 1: Comparison of main features with key literature

	Performance criterion	Investment horizon	Market model
Capponi et al. (2021)	Mean-variance with exogeneous updating of risk aversion	finite, set ex ante	discrete-time
Cui et al. (2019)	Mean-variance induced utility maximization	finite, set ex ante	discrete-time
Dai et al. (2021b)	Mean-variance for log returns	finite, set ex ante	continuous-time
This paper	m -forward process, endogeneous updating of preferences	flexible	discrete-time

The work of [Capponi et al. \(2021\)](#) is most closely related to our paper and inspired many of the ideas we will subsequently discuss. In their model, the market dynamics depends on an observable time-homogeneous Markov chain representing economic regimes. Preferences of the agent are modelled according to a multi-period mean-variance objective with a finite investment horizon. A key feature of their model is that the risk preferences of the agent are dynamic and stochastic. However, the robo-advisor cannot observe the risk preferences of the agent at all times and thus has to construct a proxy risk aversion process which is then used in the dynamic mean-variance optimization problem. Only at times when the client and robo-advisor interact will the later become aware of the idiosyncratic component of the client’s risk preferences. Since interaction times occur at a slower pace than trading times, the robo-advisor has to automatically construct a proxy for the risk preferences of the agent and trade on her behalf between two consecutive interaction times.

The setting where trading occurs at a higher frequency than performance measurement updating is reminiscent of the framework of m -forward processes we study herein and thus prompted us to explore possible applications of our results for robo-advising. Different from the setup in [Capponi et al. \(2021\)](#), we herein do not directly consider an information asymmetry about time-varying and stochastic risk preferences of the client and also ignore the possibility of behavioral biases when the client communicates her preferences. Instead of conveying the

updated idiosyncratic component of the risk preferences, the client has time-varying beliefs about the financial market and communicates those at infrequently occurring interaction times. The risk preferences are then updated endogenously in response to the updated beliefs of the client. For example, if the agent believes that the market is more volatile, then the risk-preferences adjust to this in a way conforming with Martingale Optimality Principle. Unlike in the setting of time-varying and stochastic preferences that are exogeneous, this endogenous updating of preferences assures time-consistency of optimal investment strategies.

Capponi et al. (2021) introduced a measure of portfolio personalization to analyze the following tradeoff between higher and lower interaction frequencies: When the interaction frequency is low, the robo-advisor has increasingly vague information about the current risk preferences of the agent, but when the interaction frequency is high the updated risk preferences might be subject to the behavioral biases of the agent. We herein perform a similar analysis where we seek to determine an optimal interaction schedule balancing a tradeoff between not knowing the client's current beliefs about the market parameters and an updating cost reflecting the time and effort required of the client to interact with the robo-advisor.

5.2 Optimal interaction schedule

We consider a client of a robo-advisor whose preferences are described by a discrete-time predictable forward process. At time zero, the client communicates her initial preferences, her assessment of the market parameters, and her scheduled sequence of discrete-time interaction times, corresponding to an interaction frequency $1/m$, to the robo-advisor. The robo-advisor then determines the forward utility and invests period-by-period on behalf of the agent until their next point of contact after m periods. Apparently, the framework of multi-period predictable forward processes in discrete times we have developed above is applicable because the robo-advisor receives the initial datum U_0 , the evaluation period length m , and estimation of market parameters as the model input from the client.

For the asset allocation problem faced by the robo-advisor, the updating of parameters which

determines investor's utility function and investment strategy occurs at a lower frequency than trading. It is plausible to assume that the client would update her assessment of the market parameters in each trading period upon careful contemplation, but this only reflects on her investment when interacting with the robo-advisor. We quantify the loss in performance due to infrequent interaction with the robo-advisor by comparison with a benchmark in which the agent manages the investment by herself and incorporates updated beliefs about the market parameters at each point in time. By further incorporating an updating cost which occurs at each interaction time, we aim to find the optimal updating frequency which maximizes the expected performance. The updating cost can be interpreted as the aggregation of the time and effort required of the client to interact with the robo-advisor.

Consider a client who considers two alternative interaction schedules, say A and B. Under schedule A, the client updates her assessment of the market parameters period by period, while under schedule B, the client maintains the same market parameters corresponding to her initial assessment upon the next interaction time. For simplicity, we will focus on the updating of p , the probability of an upward movement, and assume that the assessments of the possible values of the return of the stock, u and d , are constant and the same for both schemes. In the first period, the implied strategies corresponding to the two schemes are identical because the forward process looks incrementally into the future. However, in subsequent periods, the beliefs about the likelihood of an upward movement $p_i, i = 2, 3, \dots, m$ under schedule A do not necessarily coincide with the initial assessment maintained under schedule B, and the implied investment strategies thus differ until the next interaction at time m . A similar analysis holds for every evaluation period $[(k-1)m, km), k \in \mathbb{N}$.

In order to evaluate the performance of alternative interaction schedules, we consider an investment horizon T and denote by \mathbb{P}^1 the probability measure where beliefs are updated in each period. The operator \mathbb{E}_1 denotes the expectation under \mathbb{P}^1 and by $U^{(1)}$ the 1-forward process. The performance of an interaction schedule corresponding to an evaluation period length $m < T$ is given by $\mathbb{E}_1 \left[U_T^{(1)}(X_T^{(m)}) \right]$, where optimal wealth $X^{(m)}$ corresponds to the

m -forward process. In the following, we study two approaches of determining the optimal interaction schedule m^* : a robust approach and a specific example where the updating rule is described by a maximum likelihood estimator.

5.2.1 A robust criterion: maximum possible loss

Under the robust approach, we seek to derive bounds on $\mathbb{E}_1 \left[U_T^{(1)}(X_T^{(m)}) \right]$ which hold under any predictable updating rule. In other words, we allow for any predictable process $(p_i)_{i=1,\dots,T}$ which remains within a reasonable interval specified at the beginning of each evaluation period, and then compute the performance corresponding to the worst possible specification this distribution could take.

The transition probability for each trading period is given in reference to the initial probability p by $p_i = D_i p$, where D_i is a \mathcal{F}_{i-1} -measurable random variable and can take value in the interval $[D_{i,d}, D_{i,u}]$. $D_{i,d}$ and $D_{i,u}$ are some constants satisfying $0 < D_{i,d} \leq 1 \leq D_{i,u}$ and depending on the choice of interaction schedule m , thus p_i is bounded between $[D_{i,d}p, D_{i,u}p]$. To maintain absence of arbitrage, we must have $0 < p_i < 1$, and it is thus without loss of generality that D_i takes values in a bounded interval. It seems plausible to assume that $D_{i,u} - D_{i,d}$ increases over time during one evaluation period $(km, (k+1)m]$, $k \in \mathbb{N}_0$, and then resets to a smaller level at the beginning of next period after a new interaction with robo-advisors. Indeed, this behavior reflects the intuition of increasing possible deviations from the current beliefs to the original beliefs about the probability of a positive return in the time passed since the original beliefs were formed. We model this behavior by assuming periodicity on $D_{i,u}$ and $D_{i,d}$ in the interaction schedule m , i.e. $D_{i,d} = \tilde{D}_{(i \bmod m),d}$, $D_{i,u} = \tilde{D}_{(i \bmod m),u}$, for $i = 1, \dots, T$, where \bmod denotes modulo operator and the sequences $(\tilde{D}_{i,d})_{i=1}^T$ and $(\tilde{D}_{i,u})_{i=1}^T$ are exogenously given.

Under the infrequent interaction schedule B, the agent updates her beliefs about transition probability with robo-advisors at time point $0, m, 2m, \dots, km, k \in \mathbb{N}_0, km < T$, and in these periods, the robo-advisor knows the up-to-date market parameters exactly and loss through asymmetric information can thus be avoided, i.e., $\tilde{D}_{1,u} = \tilde{D}_{1,d} = 1$. In our analysis, price levels

u and d of return $R_i, i = 1, 2, \dots, m$ remain unchanged and are estimated at the beginning of the investment process. Let $a_i = \frac{1-p_i}{p_i} \frac{q}{1-q}, b = \frac{1-q}{q}, c_i = \frac{1-p_i}{1-q}, i = 1, 2, \dots, T$, and $\delta = \frac{1+b}{c_1^{\theta}(a_1^{-\theta}+b)}$.

We integrate an interaction cost into our analysis which reflects the time and effort needed to interact with the robo-advisor. We suppose that this cost is proportional to the agent's current wealth in the following sense: Whenever the agent interacts with the robo-advisor, her wealth is reduced from x to αx for some $0 < \alpha \leq 1$.

The following results are derived under the further assumption that, whenever clients interacts with the robo-advisor, the transition probability is set back to the original value p . This assumption is made to obtain a more parsimonious setting and for tractability when deriving bounds on the expected performance value under any predictable updating of beliefs.

Proposition 3. *Suppose that the initial datum is of the form $U_0(x) = (1 - \frac{1}{\theta})^{-1} x^{1-\frac{1}{\theta}}, x > 0$, for some $1 \neq \theta > 0, \theta \neq -\log_a b$, and let $T \in \mathbb{N}$ be an evaluation horizon. Let m be an interaction schedule that is a divisor of T , i.e., $m \in \mathbb{N}$ and $T/m \in \mathbb{N}$, and let $(D_i)_{i=1, \dots, T}$ be a predictable process taking values in $[D_{i,d}, D_{i,u}]$, where $D_{i,d}$ and $D_{i,u}$ satisfy the assumption of periodicity in the interaction schedule m and are such that absence of arbitrage is maintained. Then, the optimal expected performance value $\mathbb{E}_1 \left[U_T^{(1)}(X_T^{(m)}) \right]$ is bounded between*

$$\left[\alpha^{(\frac{T}{m}-1)(1-\frac{1}{\theta})} \left(\prod_{j=1}^m f_j \right)^{\frac{T}{m}} \delta^{T(1-\frac{1}{\theta})} U_0(x), \alpha^{(\frac{T}{m}-1)(1-\frac{1}{\theta})} U_0(x) \right],$$

where $1 - \alpha$ denotes the proportional interaction cost and f_j are given by $f_1 = \delta^{\frac{1}{\theta}-1}$ and $f_j = \min\{f_{D_{j,u}}, f_{D_{j,d}}\}$ if $\theta > 1$, respectively $f_j = \max\{f_{D_{j,u}}, f_{D_{j,d}}\}$ if $\theta < 1$, with

$$f_{D_{j,u}} = \frac{C_1}{\left(C_1 + \left(\frac{D_{j,u}(1-p)}{1-D_{j,u}p} \right)^{-\theta} C_2 \right)^{\frac{1}{\theta}}} + \frac{C_2}{\left(\left(\frac{D_{j,u}(1-p)}{1-D_{j,u}p} \right)^{\theta} C_1 + C_2 \right)^{\frac{1}{\theta}}},$$

$$f_{D_{j,d}} = \frac{C_1}{\left(C_1 + \left(\frac{D_{j,d}(1-p)}{1-D_{j,d}p} \right)^{-\theta} C_2 \right)^{\frac{1}{\theta}}} + \frac{C_2}{\left(\left(\frac{D_{j,d}(1-p)}{1-D_{j,d}p} \right)^{\theta} C_1 + C_2 \right)^{\frac{1}{\theta}}},$$

where $C_1 = p^\theta q^{1-\theta}$, $C_2 = (1-p)^\theta (1-q)^{1-\theta}$, and $j = 2, \dots, m$. Furthermore, f_j is non-increasing in $D_{j,u}$ and non-decreasing in $D_{j,d}$.

The following proposition shows that the optimal interaction schedule m^* is independent of the evaluation horizon T when the agent takes a robust approach of maximizing the minimal expected performance and the sequence of intervals $[D_{i,d}, D_{i,u}]$, $i = 1, 2, \dots, T$ is set to repeat itself after every updating as assumed above. This finding highlights the flexibility offered by forward performance processes to model preferences of clients of robo-advisors.

Proposition 4. *Let $(D_{i,d})_{i=1}^\infty$ and $(D_{i,u})_{i=1}^\infty$ be periodic in the interaction schedule and such that absence of arbitrage is maintained. There exists an optimal updating schedule m^* maximising the minimal expected performance for any T which is a multiple of m^* . Moreover, the optimal interaction schedule m^* can be determined by maximising the function $\mathbb{N} \rightarrow \mathbb{R}$ given by*

$$m \mapsto \left(\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^m (f_j \delta^{1-\frac{1}{\theta}}) \right)^{\frac{1}{m}}. \quad (12)$$

According to Proposition 4, the optimal interaction schedule m^* depends on the market parameters p , u , and d , the (constant) Arrow-Pratt measure of relative risk aversion $1/\theta$ of the initial datum of an agent, and the uncertainty about the evolution of future beliefs captured in the sequences $D_{i,d}$ and $D_{i,u}$, but not on the evaluation horizon T . In practice, at time zero, we choose m^* based on our current understanding of the market. At the subsequent interaction time, we update the market parameters, and then choose a new optimal interaction schedule. Therefore, market parameters, updating frequencies, preferences and investment strategies move together forward in time.

Suppose for the following discussion that $\theta > 1$, the case where $\theta < 1$ can be treated similarly. Since $f_j \delta^{1-\frac{1}{\theta}} \in (0, 1]$, the term $\prod_{j=1}^m (f_j \delta^{1-\frac{1}{\theta}})$ is decreasing in m . On the other hand, the base which belongs to $(0, 1]$ raised to the power $1/m$ is increasing in m . Hence, there are two extreme cases: First, when the rate of decline in $\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^m (f_j \delta^{1-\frac{1}{\theta}})$ is very slow, i.e., when the probability for a positive return hardly varies over different periods, then (12) is

strictly increasing in m . In this case, the strategy of never interacting with the robo-advisor is optimal, m^* tends to $+\infty$. Second, when the rate of decline in $\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^m (f_j \delta^{1-\frac{1}{\theta}})$ is very fast, i.e., when p_i changes substantially across periods and the updating cost is small, then (12) is strictly decreasing in m . In this case, the strategy of period-by-period updating is optimal, i.e., $m^* = 1$. As we discussed earlier, the rate of decline is typically slow at first and then increases as more and more time elapsed since the last interaction time as a consequence of the increasing width $D_{i,u} - D_{i,d}$. Also, the rate of decline in $1/m$ is strictly decreasing, which means the degree of growth resulted from the decreasing exponent is weakening as m increases. If this is the case, we are typically able to determine a unique optimal interaction time m^* which is larger than one.

Intuitively, m^* is increasing in the interaction cost and decreasing in the uncertainty about parameters. These are the two competing forces in our model, and m^* attempts to find an ideal balance between them. In the following, we will confirm this intuition. We retain the assumption that the sequence of intervals $[D_{i,d}, D_{i,u}]$ is periodic and consider the case where $\theta > 1$.

First, from the above analysis one can directly infer that m^* is increasing in the interaction cost. Indeed, when α decreases, the rate of decline in $\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^m (f_j \delta^{1-\frac{1}{\theta}})$ is slower and the optimal m^* that maximises (12) will thus be larger. This implies that communication between the agent and robo-advisors should be reduced if it comes at a high cost.

Second, m^* is typically decreasing in the uncertainty about parameters. In other words, one should update more frequently when there is a larger range of possible values for the transition probability, while it is better to update less frequently when the parameter is stable and we can estimate it with more confidence. To substantiate this intuition, we consider a uniform increase of uncertainty and approximate it by the case where all factors $f_j, j = 2, \dots, m$ simultaneously decrease to $f'_j = C f_j, j = 2, \dots, m$, with the same constant $C < 1$, but $f'_1 = f_1$. The rate of decline in $\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^m (f'_j \delta^{1-\frac{1}{\theta}})$ becomes quicker than $\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^m (f_j \delta^{1-\frac{1}{\theta}})$, hence m^* that maximises (12) will be smaller. Moreover, for any two alternative interaction schedules $m_1 < m_2$ and m_1

outperforms m_2 , it then holds that

$$\left(\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^{m_1} (f_j \delta^{1-\frac{1}{\theta}}) \right)^{\frac{1}{m_1}} > \left(\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^{m_2} (f_j \delta^{1-\frac{1}{\theta}}) \right)^{\frac{1}{m_2}}. \quad (13)$$

Since $\frac{1}{m_1} > \frac{1}{m_2}$ and $C^{1-\frac{1}{m_1}} > C^{1-\frac{1}{m_2}}$, we have

$$\begin{aligned} \left(\alpha^{1-\frac{1}{\theta}} C^{m_1-1} \prod_{j=1}^{m_1} (f_j \delta^{1-\frac{1}{\theta}}) \right)^{\frac{1}{m_1}} &= C^{1-\frac{1}{m_1}} \left(\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^{m_1} (f_j \delta^{1-\frac{1}{\theta}}) \right)^{\frac{1}{m_1}} \\ &> C^{1-\frac{1}{m_2}} \left(\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^{m_2} (f_j \delta^{1-\frac{1}{\theta}}) \right)^{\frac{1}{m_2}} \\ &= \left(\alpha^{1-\frac{1}{\theta}} C^{m_2-1} \prod_{j=1}^{m_2} (f_j \delta^{1-\frac{1}{\theta}}) \right)^{\frac{1}{m_2}}, \end{aligned}$$

which means that an infrequent schedule m_2 cannot perform better than m_1 when uncertainty increases uniformly.

However, one needs to be more careful when the increase of uncertainty is not uniform. For example, suppose that only the k 'th interval after every updating becomes wider, $D'_{k,u} - D'_{k,d} > D_{k,u} - D_{k,d}$, and all other parameters remain constant. The original f_k is reduced to a smaller f'_k by the last statement of Proposition 3. We investigate whether m_2 can outperform m_1 after the increase in three distinct cases: First, when $k > m_2$, (13) is not affected. Second, when $m_1 < k \leq m_2$, the performance of schedule m_1 does not change, while the performance of schedule m_2 decreases, and thus m_1 still leads to a better performance. However, in the third case: $k \leq m_1$, we might have the opposite inequality, i.e., m_2 outperforms m_1 . This happens because with lower interaction frequency, we can have less updating cost when its effect on the minimal expected performance is significant. More importantly, if $D'_{k,u}$ becomes closer to $D_{k+1,u}$ such that the benefit one can get from the updating is small, then there is no need to update immediately. Therefore, we should not only just focus on the increasing loss incurred from deviation from the actual parameter, but also take the updating frequency and cost into

consideration.

We close this section with two numerical examples. The market parameters of the first example for below table are given by $p = 0.6$, $u = 1.2$, $d = 0.8$, the constant relative risk aversion of the initial datum is $\theta = 5$, the updating cost is 0.2%, and the bounds for the updating of future probabilities are given by $D_{i,d} = D_d^{\text{mod}(i-1,m)}$, $D_{i,u} = D_u^{\text{mod}(i-1,m)}$, $i \in \{1, 2, \dots, T\}$, where $D_d = 0.99$, $D_u = 1.01$.

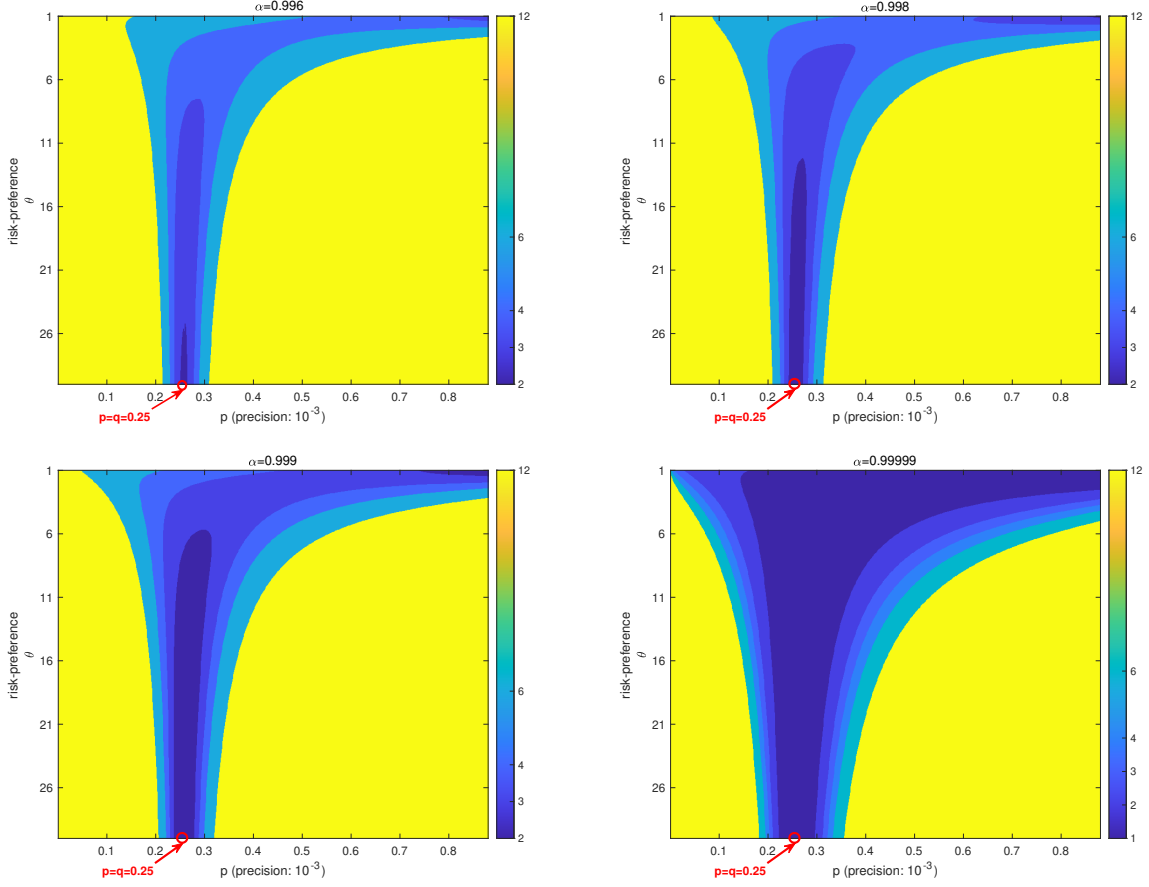
Table 2: Minimal expected performance at time T for different interaction schedules

$T = 12$	m	1	2	3	4	6	12				
	MEP	0.98	0.991	0.993	0.991	0.98	0.93				
$T = 24$	m	1	2	3	4	6	8	12	24		
	MEP	0.96	0.981	0.984	0.981	0.96	0.94	0.85	0.48		
$T = 48$	m	1	2	3	4	6	8	12	16	24	48
	MEP	0.93	0.96	0.97	0.96	0.93	0.88	0.73	0.55	0.23	0.003

Notes. The MEP (minimal expected performance) presented above is re-scaled by dividing it by $U_0(x)$. We only update at frequencies $1/m$ where m is a divisor of T .

Conforming with Proposition 4, Table 2 shows that the optimal interaction frequencies are independent of the evaluation horizon T . While the minimal expected performance, MEP in the table, is decreasing over T because losses in expected performance from both interaction and not updating timely are accumulating as time elapses, the optimal interaction schedule m^* exists and is universal for any evaluation horizon T . We can also infer that the minimal expected performance is first increasing and then decreasing with respect to m . This demonstrates the tradeoff between updating cost and deviation from actual parameter due to not interacting with robo-advisors in time.

In the second example, we investigate the impact of increasing risk aversion on the optimal interaction schedule m^* . There are two distinct cases. Figure below visualizes how the optimal interaction schedule m^* depends on the client's risk preference parameter $\theta > 1$ and the difference between p and q when $D_d = 0.99$, $D_u = 1.01$.


 Figure 1: Optimal interaction schedule m^* computed with different market parameters

In the first case, when $|p - q|$ or θ is large enough, we observe that a more risk-averse agent is interacting more frequently with the robo-advisor than a less risk-averse agent.

However, in the second case where p is close to q , or when the agent is already extremely risk-averse, we make the opposite observation that the agent decreases her interaction frequency as she becomes more risk-averse. This is because, in this case, the investment in the risky asset is very small, and the updating of the probability for a positive outcome does not lead to a significant change in optimal investment strategies. This situation is especially likely to occur when, at the same time, the updating cost plays a relevantly important role in determining the optimal interaction schedule. Furthermore, we observe from all the first three heat maps that, as θ increases, or equivalently risk aversion decreases, the region where the optimal interaction schedule increases as the agent becomes more risk averse becomes narrower around the region

where $p = q$. The influence of the interaction cost on the width of this region depends on two competing forces. First, since the agent prefers interacting less frequently when faced with higher interaction cost as argued above, the set of (θ, p) -combinations leading to an optimal interaction schedule m^* that is smaller than the evaluation horizon (the areas of a color other than yellow in the heat map) are reduced to an increasingly narrow band around the value $p = q$ as interaction cost increases. However, the width might also become larger as the interaction cost grows in situations where the loss from each interaction outweighs the benefit of accurate knowledge about the model parameters. Therefore, the region where the optimal interaction schedule increases as the agent becomes more risk averse does not grow monotonically in the interaction cost as shown in Figure 1.

5.2.2 An explicit updating rule: Maximum likelihood estimator for positive return probability

We study an explicit example where the probabilities for a positive return of the stock $p_i, i \in \mathbb{N}$ are the maximum likelihood estimators given past information. Specifically, suppose that there are N observations about the performances of risky asset at time zero, and that the stock achieved a positive return N_u times. The maximum likelihood estimator for an upward move of the stock in the first period $[0, 1)$ is thus given by $p_1 = \frac{N_u}{N}$, in the second period $[1, 2)$ by

$$p_2 = \frac{Np_1 + 1}{N + 1} \mathbb{1}_{\{R_1=u\}} + \frac{Np_1}{N + 1} \mathbb{1}_{\{R_1=d\}} \quad (14)$$

and so forth. Let N_t^u represent the process of total number of positive returns of the stock from time 0 until time t starting from $N_0^u = 0$. We then have for $t = 1, 2, 3, \dots, m - 1$,

$$p_{t+1} = \frac{Np_1 + N_t^u}{N + t}, \quad 1 - p_{t+1} = \frac{N(1 - p_1) + t - N_t^u}{N + t}. \quad (15)$$

As in the previous section, we seek to determine an interaction schedule that represents an optimal trade-off between loss in performance value due to the deviation from the actual

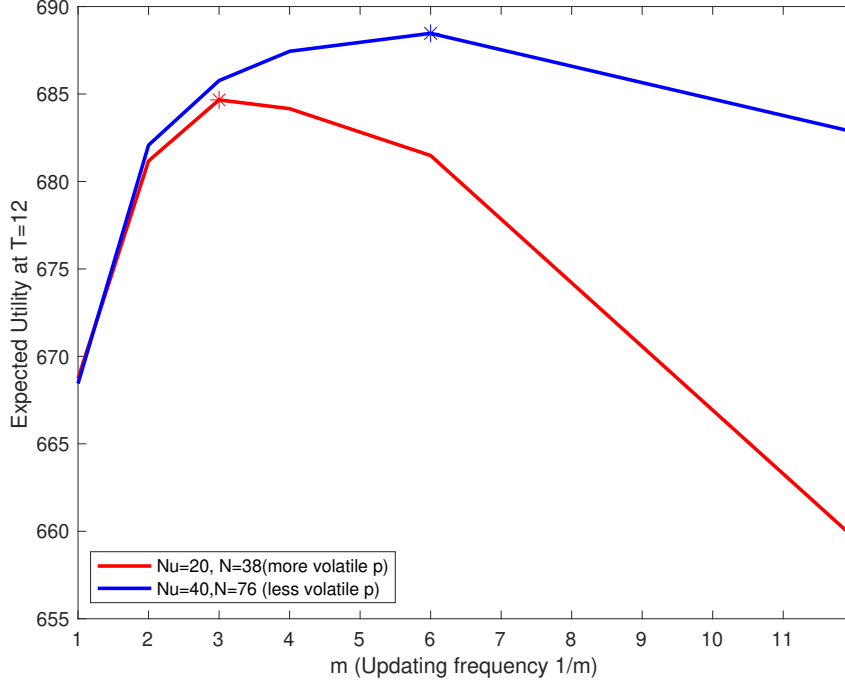


Figure 2: Numerical example of finding optimal updating frequency for terminal expected utility $\mathbb{E}_1[U_T^{(1)}(X_T^{(m)})]$ at $T = 12$

assessment of the market and the updating cost occurring when interacting with the robo-advisor. We limit our analysis on a numerical example where we compare two settings where the initial assessments of the p_1 are identical, but one is based on more observations than the other. The parameter values for this example are $u = 1.3$, $d = 0.8$, $\theta = 3$, $m \in \{1, 2, 3, 4, 6, 12\}$ which are the factors of $T = 12$. We again consider an initial utility function of the form $U_0(x) = (1 - \frac{1}{\theta})^{-1} x^{1-\frac{1}{\theta}}$, $x > 0$, a proportional interaction cost set to $\alpha = 0.4\%$, and suppose that the initial wealth is $X_0 = 9960$ corresponding to an initial wealth of 10000 minus the interaction cost. We perform 10^8 simulations to compute all involved expected values.

Figure 2 shows the optimal interaction schedule m^* at which the expected performance is maximal. We observe that the expected performance is first increasing and then decreasing as a function of the interaction schedule m . This is what we expect: On the one hand, if the client interacts with the robo-advisor too frequently, the loss due to the interaction cost dominates, but, on the other hand, if there are too few interactions and updates in beliefs are

not communicated to the robo-advisor in a timely manner, the loss due to inaccurate model parameters dominates.

The blue and red scenarios correspond to settings where we have more (blue), respectively less (red), prior observations of the stock performance. When we have a large number of prior observations, our assessment of the probability of an upward movement is less susceptible to a single new information than when we have fewer observations. This translates to a larger interaction schedule m being optimal, since it becomes less important to immediately communicate the updated assesment of the market to the robo-advisor.

6 Conclusions

We have studied discrete-time predictable forward processes when trading dates do not coincide with performance evaluation dates in a binomial tree model for the financial market. Our main technical contributions are conditions for existence and explicit solutions for the functional equations associated with the construction of predictable forward processes. We have then applied the obtained results to study the asset allocation problem faced by robo-advisors, an application where performance evaluation naturally occurs at a lower frequency than trading. Our findings and discussions show that predictable forward performance processes constitute a viable framework to model preferences of agents of robo-advisors and can provide valuable insights when determining an optimal interaction schedule between the robo-advisor and its human clients.

Appendix. Proofs

A Proof of Theorem 1

In order to prove Theorem 1, we start by first presenting a result which shows how to reduce the functional equation (4) associated with the inverse optimization problem (2) to a system

of linear functional equations of order one. These simpler equations can then be solved using existing techniques.

Theorem 3. *Let I_0 and I_m be related by (4) and suppose that the market is homogeneous. Then there exist functions $(\psi_i)_{i=0}^{m-1}$ solving the system of linear functional equations of order one*

$$\psi_{m-1}(ay) + b\psi_{m-1}(y) = (1+b)^m I_0(c^m y), \quad (16)$$

$$\psi_{i-1}(ay) + b\psi_{i-1}(y) = \psi_i(y), \quad i = m-1, m-2, \dots, 1, \quad (17)$$

with $\psi_0 = I_m$. On the other hand, if $(\psi_i)_{i=0}^{m-1}$ is a family of functions satisfying (16) and (17) for a given I_0 , then, by defining $I_m = \psi_0$, the pair (I_0, I_m) satisfies (4).

Proof. When $p = q$, we obviously have $I_m(y) = I_0(y)$ as the unique solution to (4). Since in this case $a = c = 1$, the functions $\psi_i(y) = (1+b)^i I_0(y)$, $i = 0, 1, \dots, m-1$ solve the system (16) and (17) with $\psi_0(y) = I_0(y) = I_m(y)$. On the other hand, any $(\psi_i)_{i=0}^{m-1}$ satisfying (16) and (17) must also satisfy $\psi_i(y) = (1+b)^i I_0(y)$, $i = 0, 1, \dots, m-1$ for a given I_0 , and $I_m(y) = \psi_0(y)$ solves (4).

Henceforth, we assume that $p \neq q$. After making change of variable $\hat{y} = c^m y$, multiplying both sides with q^{-m} , and recalling that $a = \frac{1-p}{p} \frac{q}{1-q}$, $b = \frac{1-q}{q}$, and $c = \frac{1-p}{1-q}$, (4) becomes

$$(1+b)^m I_0(c^m y) = \sum_{i=0}^m \binom{m}{i} b^{m-i} I_m(a^i y). \quad (18)$$

This is a linear functional equation of order m . We follow the method of reduction of order, see, e.g., (Kuczma et al., 1990, Section 6.7), to reduce this equation to a system of linear functional equations of order one. To this end, we let

$$\psi_1(y) := I_m(ay) - \lambda_0 I_m(y)$$

where λ_0 is to be determined. We get by induction for $i = 1, \dots, m$ that

$$I_m(a^i y) = \psi_1(a^{i-1} y) + \sum_{k=1}^{i-1} \lambda_0^{i-k} \psi_1(a^{k-1} y) + \lambda_0^i I_m(y).$$

Inserting this expression into (18), we obtain

$$\begin{aligned} (1+b)^m I_0(c^m y) &= \sum_{i=1}^m \binom{m}{i} b^{m-i} \left(\psi_1(a^{i-1} y) + \sum_{k=1}^{i-1} \lambda_0^{i-k} \psi_1(a^{k-1} y) \right) \\ &\quad + \sum_{i=0}^m \binom{m}{i} b^{m-i} \lambda_0^i I_m(y). \end{aligned}$$

By virtue of the Binomial Theorem, $\sum_{i=0}^m \binom{m}{i} b^{m-i} \lambda_0^i = (b + \lambda_0)^m$. Setting $\lambda_0 = -b$ thus yields

$$(1+b)^m I_0(c^m y) = \sum_{i=1}^m \binom{m}{i} b^{m-i} \left(\psi_1(a^{i-1} y) + \sum_{k=1}^{i-1} (-b)^{i-k} \psi_1(a^{k-1} y) \right), \quad (19)$$

which is a linear functional equation for ψ_1 of order $m-1$. Note that the coefficient of any term $\psi_1(a^j y)$, $j = 0, \dots, m-1$ in (19) is given by $b^{m-1-j} \sum_{n=j+1}^m (-1)^{n-(j+1)} \binom{m}{n}$. By the recursive formula for binomial coefficients, $\binom{m}{n} = \binom{m-1}{n-1} + \binom{m-1}{n}$, we further have the relationship $\sum_{n=j+1}^m (-1)^{n-(j+1)} \binom{m}{n} = \binom{m-1}{j}$. Hence, (19) can be simplified to

$$(1+b)^m I_0(c^m y) = \sum_{j=0}^{m-1} \binom{m-1}{j} b^{m-1-j} \psi_1(a^j y).$$

By sequentially defining $\psi_i(y) = \psi_{i-1}(ay) + b\psi_{i-1}(y)$, $i = 2, \dots, m-1$, we can obtain $(1+b)^m I_0(c^m y) = \sum_{j=0}^{m-i+1} \binom{m-i+1}{j} b^{m-i+1-j} \psi_{i-1}(a^j y)$ through an exact repetition of the steps outlined above. Finally we can reduce (19) to (16), an equation for ψ_{m-1} of order 1. To summarize, we obtained a solution to the system of linear functional equations of order one in (16) and (17) as claimed in the first part of the theorem.

Reversing the above calculations directly shows other direction of the theorem. Indeed, starting from (16) and sequentially replacing $\psi_i(y)$ with $\psi_{i-1}(ay) + b\psi_{i-1}(y)$ shows that $I_m = \psi_0$ solves (18) and thus also (4). \square

Remark 2. *Theorem 3 holds for general solutions I_0 and I_m which are not necessarily inverse marginal functions. However, in this paper, we will apply the theorem to solutions I_0 and I_m of (4) in the class of inverse marginal functions.*

Linear functional equations of the form (16) and (17) typically have multiple solutions, even when one restricts admissible functions to remain within the class of inverse marginal functions, cf. (Angoshtari et al., 2020, Example 6.1) for an illustrative example. However, it is possible to obtain a unique solution in a smaller class of functions restricting the behavior as the argument goes to zero or infinity. The following auxiliary result providing general uniqueness conditions for linear functional equations of order one is a version of (Angoshtari et al., 2020, Lemma 6.2) adapted to our notation.

Lemma 1 (Angoshtari et al. (2020)). *Let I_0 respectively ψ_i be given. Then, there exists at most one solution ψ_{i-1} to (16) or (17), satisfying $\lim_{y \rightarrow 0^+} y^{-\log_a b} \psi_{i-1}(y) = 0$. Similarly, there exists at most one solution satisfying $\lim_{y \rightarrow \infty} y^{-\log_a b} \psi_{i-1}(y) = 0$.*

We next recall another main result (Angoshtari et al., 2020, Theorem 6.3) providing sufficient conditions for existence and uniqueness of solutions to equations of the form (16) and (17) in the class of inverse marginal functions.

Lemma 2 (Angoshtari et al. (2020)). *Let $i \in \{1, \dots, m\}$ be fixed and let I_0 respectively ψ_i be a given inverse marginal function. Define the functions*

$$\begin{aligned} \Phi(y) &= I_0(ac^m y) - bI_0(c^m y) \quad \text{and} \quad \Psi(y) = y^{-\log_a b} I_0(c^m y), \quad y > 0, \quad \text{when } i = m, \\ \Phi(y) &= \psi_i(ay) - b\psi_i(y) \quad \text{and} \quad \Psi(y) = y^{-\log_a b} \psi_i(y), \quad y > 0, \quad \text{otherwise.} \end{aligned} \tag{20}$$

The following assertions hold:

(i) *If (Φ, Ψ) satisfies condition (C1), then a solution ψ_{i-1} is given by*

$$\begin{aligned} \psi_{m-1}(y) &= \frac{(1+b)^m}{b} \sum_{n=0}^{\infty} (-1)^n b^{-n} I_0(a^n c^m y), \quad \text{when } i = m; \\ \psi_{i-1}(y) &= \frac{1}{b} \sum_{n=0}^{\infty} (-1)^n b^{-n} \psi_i(a^n y), \quad \text{otherwise.} \end{aligned}$$

(ii) If (Φ, Ψ) satisfies condition (C2), then a solution is given by

$$\begin{aligned}\psi_{m-1}(y) &= (1+b)^m \sum_{n=0}^{\infty} (-1)^n b^n I_0(a^{-(n+1)} c^m y), \quad \text{when } i = m; \\ \psi_{i-1}(y) &= \sum_{n=0}^{\infty} (-1)^n b^n \psi_i(a^{-(n+1)} y), \quad \text{otherwise.}\end{aligned}$$

(iii) In parts (i) and (ii), the corresponding ψ_{i-1} satisfies the uniqueness conditions of Lemma 1 and, moreover, $\psi_{i-1} \in \mathcal{I}$.

(iv) The solution ψ_{i-1} given in (i) and (ii), respectively, is the only positive solution and thus also the only inverse marginal function to the corresponding equation (16) or (17).

We now have all the ingredients required for the proof of Theorem 1.

Proof of Theorem 1. As discussed in Section 2, if two utility functions U_0 and U_m solve problem (2), then their associated inverse marginals satisfy (4). Conversely, when a pair of inverse marginal functions I_0 and I_m satisfy (4), then the corresponding utility functions satisfy (2) up to a constant. Theorem 2.4 in Strub and Zhou (2021) together with the subsequent discussion therein shows that U_m defined as in Theorem 1 does indeed solve (2) when I_m solves (4). Moreover, the expression for the optimal wealth X_m^* follows from the existing theory on classical expected utility maximization once we obtained U_m and regard (2) as a classical, backward problem. Therefore, it remains to show that I_m given in (7) under the assumption that $(A_i)_{i=0,\dots,m}$ exists is the unique solution to (4) in the class of inverse marginal functions.

By Theorem 3, we can reduce (2) to the system of linear functional equations of order one in (16) and (17). We solve this system by iteratively applying Lemma 2. Specifically, we will show by induction that the solutions $(\psi_i)_{i=0}^{m-1}$ to (16) and (17) must be given by

$$\psi_{m-i}(y) = \frac{(1+b)^m}{b^{A_i}} \sum_{n_1=0, \dots, n_i=0}^{\infty} (-1)^{p(n_1, \dots, n_i)} b^{q_{A_i}(n_1, \dots, n_i)} I_0(a^{r_{A_i}(n_1, \dots, n_i)} c^m y), \quad (21)$$

and that the functions Φ_i and Ψ_i defined in (20) for deriving ψ_{m-i-1} from ψ_{m-i} indeed correspond to $\Phi_i^{A_i}$ and $\Psi_i^{A_i}$ given in (6).

To show the base case, we consider (16) and, according to Lemma 2, have

$$\psi_{m-1}(y) = \begin{cases} \frac{(1+b)^m}{b} \sum_{n=0}^{\infty} (-1)^n b^{-n} I_0(a^n c^m y), & \text{if } (\Phi_0^0, \Psi_0^0) \text{ satisfy (C1),} \\ (1+b)^m \sum_{n=0}^{\infty} (-1)^n b^n I_0(a^{-(n+1)} c^m y), & \text{if } (\Phi_0^0, \Psi_0^0) \text{ satisfy (C2).} \end{cases} \quad (22)$$

Noticing that in the first case $A_1 = A_0 + 1 = 1$ while in the second case $A_1 = A_0 = 0$ shows the claim. As a general inductive step of deriving ψ_{m-j-1} under the assumption that ψ_{m-j} is given by (21) and we have already obtained A_j , we solve equation (17) for $i = m - j$. Firstly, we define Φ_j and Ψ_j according to (20) which is then consistent with (6) with ψ_i given by (21), and apply Lemma 2 to obtain that if (Φ_j, Ψ_j) satisfies (C1), $\psi_{m-j-1}(y)$ is given by

$$\psi_{m-j-1}(y) = \frac{(1+b)^m}{b^{A_j+1}} \sum_{n_1=0, \dots, n_{j+1}=0}^{\infty} (-1)^{p(n_1, \dots, n_{j+1})} b^{q_{A_j; (n_1, \dots, n_j)} - n_{j+1}} I_0(a^{r_{A_j; (n_1, \dots, n_j)} + n_{j+1}} c^m y),$$

and if (Φ_j, Ψ_j) satisfies (C2), $\psi_{m-j-1}(y)$ is given by

$$\psi_{m-j-1}(y) = \frac{(1+b)^m}{b^{A_j}} \sum_{n_1=0, \dots, n_{j+1}=0}^{\infty} \left((-1)^{p(n_1, \dots, n_{j+1})} b^{q_{A_j; (n_1, \dots, n_j)} + n_{j+1}} \times I_0(a^{r_{A_j; (n_1, \dots, n_j)} - (n_{j+1}+1)} c^m y) \right).$$

Therefore, $\psi_{m-j-1}(y)$ can be expressed by (21) with $A_{j+1} = A_j + 1$ when (C1) is satisfied or $A_{j+1} = A_j$ when (C2) is satisfied. This shows the claim.

Finally we conclude that the inverse marginal $\psi_0 = I_m$ can be expressed by (7). Since we need to ensure that I_m is strictly positive as an inverse marginal, by the relationship between ψ_{i-1} and ψ_i in (17), $\psi_j, j = 1, 2, \dots, m-1$ must all be strictly positive. These solutions are thus unique within the class of inverse marginal functions by part (iv) of Lemma 2. We conclude that I_m given in (7) is the unique inverse marginal solving (4) if $(A_i)_{i=0, \dots, m}$ exists. \square

B Proof of Corollary 1

Let $x > 0$. We first note that a, b , and c are all Borel-measurable functions of the market parameters $(p, u, d) \in \mathcal{M}$. We will drop the classifier Borel- for the remainder of this proof. Next, we prove the measurability of $(A_i)_{i=0, \dots, m}$ by induction. The base case, $i = 0$, is trivial since $A_0 = 0$ is constant. We then assume that A_i is measurable, and prove measurability of A_{i+1} . Recall that A_{i+1} can be expressed as

$$A_{i+1} = \sum_{j=0}^i \mathbb{1}_{\{A_i=j\}} \left[(j+1) \mathbb{1}_{\{(p,u,d) \in \mathcal{M}: (\Phi_i^j, \Psi_i^j) \text{ satisfies (C1)}\}} + j \mathbb{1}_{\{(p,u,d) \in \mathcal{M}: (\Phi_i^j, \Psi_i^j) \text{ satisfies (C2)}\}} \right].$$

By Lemma 2 and since I_0 is continuously differentiable, the infinite series of $(\Phi_i^j)'$ and Ψ_i^j converge for $(p, u, d) \in \mathcal{M}$. Therefore, $(\Phi_i^j)'$ and Ψ_i^j defined in (6) are measurable as pointwise limits of measurable functions. Hence, the two functions $\mathbb{1}_{\{(p,u,d) \in \mathcal{M}: (\Phi_i^j, \Psi_i^j) \text{ satisfies (C1)}\}}$ and $\mathbb{1}_{\{(p,u,d) \in \mathcal{M}: (\Phi_i^j, \Psi_i^j) \text{ satisfies (C2)}\}}$ are measurable, which in turn shows that A_{i+1} is measurable.

Note that the series of I_m is derived by sequential application of Lemma 2. Since all intermediate functions are shown to be convergent, so does I_m . The measurable dependence of I_m on the market parameters then follows from the explicit expression in (7) as a pointwise limit of measurable function in a converging series.

In a multi-period binomial market, the expectation in the expression of U_m is essentially a finite sum of integral terms. Given that the inverse function of a strictly monotone function is also strictly monotone and thus integrable over finite intervals, its corresponding integral with variable lower limit of integration $f(u) = \int_u^x I_m^{-1}(t) dt$ for any given x exists and is continuous. Therefore, the integral terms $f(I_m(\frac{d\mathbb{Q}}{d\mathbb{P}} U_0'(1)))$, are compositions of two measurable functions, which then shows that $U_m(x)$ is a measurable function of the market parameters as claimed. \square

C Proof of Proposition 1

By virtue of Theorem 1, U_m exists and is unique. The single-period forward process \tilde{U}_i exists for $i = 0, \dots, m$ because we can define it as the value function corresponding to the backward

expected utility maximization problem with utility U_m . The required properties then follow from the standard theory on stochastic control. On the other hand, if \tilde{U}_i , $i = 0, \dots, m$, is a single-period forward process with $\tilde{U}_0 = U_0$, with associated optimal wealth process \tilde{X}^* , then

$$U_0(x) = \mathbb{E} [\tilde{U}_1(\tilde{X}_1^*)] = \mathbb{E} [\mathbb{E} [\tilde{U}_2(\tilde{X}_2^*) | \mathcal{F}_1]] = \dots = \mathbb{E} [\tilde{U}_m(\tilde{X}_m^*)]$$

and, with a similar argument, $U_0(x) \geq \mathbb{E} [\tilde{U}_m(\tilde{X}_m)]$ for any $\tilde{X} \in \mathcal{X}(x)$. Therefore, (U_0, \tilde{U}_m) is an m -forward pair and, by uniqueness established in Theorem 1, $U_m = \tilde{U}_m$. \square

D Proof of Proposition 2

Let $m \in \mathbb{N}$ and $x > 0$ be given. Without loss of generality, we discuss the time-monotonicity between forward utilities $U_m(x)$ and $U_{m+1}(x)$, $x \in \mathbb{R}^+$. By virtue of Proposition 1, the perid-by-period forward process \tilde{U} exists uniquely and satisfies $\tilde{U}_m(x) = U_m(x)$ and $\tilde{U}_{m+1}(x) = U_{m+1}(x)$. Therefore, the forward utility U_m is the value function corresponding to a one-period backward expected utility maximization problem with objective function U_{m+1} , i.e., U_m and U_{m+1} satisfy the relationship $U_m(x) = \text{ess sup}_{X_{m+1} \in \mathcal{X}(m, x)} \mathbb{E} [U_{m+1}(X_{m+1}) | \mathcal{F}_m]$.

The supremum is essentially taken over all investment strategies in the market and thus should be larger than the expected performance corresponding to a particular strategy: putting all the wealth in risk-free asset, i.e., $\text{ess sup}_{X_{m+1} \in \mathcal{X}(m, x)} \mathbb{E} [U_{m+1}(X_{m+1}) | \mathcal{F}_m] \geq \mathbb{E} [U_{m+1}(x)]$. Since there is no randomness in this choice, $U_m(x) \geq U_{m+1}(x)$ follows, and we have a strict inequality when the market is offering non-zero expected excess return due to the well-known fact that the optimal strategy invests a non-zero amount into the risky asset in this case. \square

E Proof of Theorem 2

Making the substitution $\hat{y} = \prod_{i=1}^m \frac{(1-p_i)}{(1-q_i)} y$ allows us to transform (8) to

$$I_0 \left(\prod_{i=1}^m \frac{(1-p_i)}{(1-q_i)} y \right) = \sum_{j=0}^{2^m-1} \prod_{i=1}^m q_i^{\gamma_{j,i}} (1-q_i)^{1-\gamma_{j,i}} I_m \left(\prod_{i=1}^m \frac{q_i^{\gamma_{j,i}} (1-p_i)^{\gamma_{j,i}}}{p_i^{\gamma_{j,i}} (1-q_i)^{\gamma_{j,i}}} y \right).$$

Next, we multiply both sides by $\left(\prod_{i=1}^m q_i\right)^{-1}$ and recall the expression in terms of $a_i, b_i, c_i, i = 1, 2, \dots, m$ to obtain

$$\prod_{i=1}^m (1 + b_i) I_0 \left(\prod_{n=1}^m c_n y \right) = \sum_{j=0}^{2^m-1} \prod_{i=1}^m b_i^{1-\gamma_{j,i}} I_m \left(\prod_{k=1}^m a_k^{\gamma_{j,k}} y \right). \quad (23)$$

Different from the homogeneous setting, the arguments of I_m are not in the form of iterate functions. We therefore cannot use the same approach as in the proof of Theorem 1. Instead, we aim to show by mathematical induction that if there exist functions $(\tilde{I}_i)_{i=1}^m$ such that they satisfy a system of equations $\tilde{I}_i(a_i y) + b_i \tilde{I}_i(y) = (1 + b_i) \tilde{I}_{i-1}(c_i y), i = 1, 2, \dots, m$ for a given I_0 , then I_0 and $I_m = \tilde{I}_m$ satisfy (23) in the heterogeneous market.

Firstly, when $m = 1$, the statement naturally holds. Let us then assume that the statement is true for $m = M$. When $m = M + 1$, the left hand side of equation (23) becomes

$$\begin{aligned} & \prod_{i=1}^{M+1} (1 + b_i) I_0 \left(\prod_{n=1}^{M+1} c_n y \right) \\ &= (1 + b_{M+1}) \prod_{i=1}^M (1 + b_i) I_0 \left(\prod_{n=1}^M c_n c_{M+1} y \right) \\ &= (1 + b_{M+1}) \sum_{j=0}^{2^M-1} \prod_{i=1}^M b_i^{1-\gamma_{j,i}} \tilde{I}_M \left(\prod_{k=1}^M a_k^{\gamma_{j,k}} c_{M+1} y \right) \\ &= \sum_{j=0}^{2^M-1} \prod_{i=1}^M b_i^{1-\gamma_{j,i}} \left(\tilde{I}_{M+1}(a_{M+1} \prod_{k=1}^M a_k^{\gamma_{j,k}} y) + b_{M+1} \tilde{I}_{M+1}(\prod_{k=1}^M a_k^{\gamma_{j,k}} y) \right). \end{aligned} \quad (24)$$

Note that the right hand side of (23) is given by $\sum_{j=0}^{2^{M+1}-1} \prod_{i=1}^{M+1} b_i^{1-\gamma_{j,i}} I_{M+1} \left(\prod_{k=1}^{M+1} a_k^{\gamma_{j,k}} y \right)$ when setting $m = M + 1$. Observe that for $j \in \{0, 1, \dots, 2^M - 1\}$, we have $\gamma_{j,M+1} = 0$, where $\gamma_{j,M+1}$ is the first digit of j in binary if expressed in $M + 1$ digits in total. Thus, the term inside the summation becomes $b_{M+1} \prod_{i=1}^M b_i^{1-\gamma_{j,i}} I_{M+1} \left(\prod_{k=1}^M a_k^{\gamma_{j,k}} y \right)$. However, for $j \in \{2^M, 2^M + 1, \dots, 2^{M+1} - 1\}$, $\gamma_{j,M+1} = 1$, and the term inside the summation is given by $\prod_{i=1}^M b_i^{1-\gamma_{j,i}} I_{M+1} \left(a_{M+1} \prod_{k=1}^M a_k^{\gamma_{j,k}} y \right)$. Hence, the right hand side of (23) is equal to the last line of (24) by letting $I_m = \tilde{I}_m$. This proves the claim for $m = M + 1$, and thus for arbitrary m by induction.

For the other direction, by the fact that with substitution $\tilde{I}_{k-1}(y) = \frac{1}{1+b_k}(\tilde{I}_k(\frac{a_k y}{c_k}) + b_k \tilde{I}_k(\frac{y}{c_k}))$, $k = m, \dots, 1$, equation

$$\prod_{i=1}^k (1 + b_i) I_0 \left(\prod_{n=1}^k c_n y \right) = \sum_{j=0}^{2^k-1} \prod_{i=1}^k b_i^{1-\gamma_{j,i}} \tilde{I}_k \left(\prod_{n=1}^k a_n^{\gamma_{j,n}} y \right)$$

will become

$$\prod_{i=1}^{k-1} (1 + b_i) I_0 \left(\prod_{n=1}^{k-1} c_n y \right) = \sum_{j=0}^{2^{k-1}-1} \prod_{i=1}^{k-1} b_i^{1-\gamma_{j,i}} \tilde{I}_{k-1} \left(\prod_{n=1}^{k-1} a_n^{\gamma_{j,n}} y \right),$$

we can show that if I_0 and I_m are related by (23), then there exist functions $(\tilde{I}_i)_{i=1}^{m-1}$ such that they satisfy a system of equations $\tilde{I}_i(a_i y) + b_i \tilde{I}_i(y) = (1 + b_i) \tilde{I}_{i-1}(c_i y)$, $i = 1, 2, \dots, m$ with $\tilde{I}_0 = I_0$ and $\tilde{I}_m = I_m$.

Solving $\tilde{I}_i(a_i y) + b_i \tilde{I}_i(y) = (1 + b_i) \tilde{I}_{i-1}(c_i y)$, $i = 1, 2, \dots, m$ by sequentially and repeatedly applying Lemma 2, which is a version of Theorem 6.3 in Angoshtari et al. (2020) with slight difference, we can finally derive the expression of I_m and obtain the conditions for uniqueness.

As in the proof of Theorem 1, we can show by induction that \tilde{I}_i , $i = 1, 2, \dots, m$ solving $\tilde{I}_i(a_i y) + b_i \tilde{I}_i(y) = (1 + b_i) \tilde{I}_{i-1}(c_i y)$, $i = 1, 2, \dots, m$, must be given by

$$\tilde{I}_i(y) = \frac{\prod_{v=1}^i (1 + b_v)}{\prod_{j=1}^i b_j^{\alpha_j}} \sum_{n_1=0, \dots, n_i=0}^{\infty} (-1)^{p(n_1, \dots, n_i)} \prod_{k=1}^i b_k^{n_k(1-2\alpha_k)} I_0 \left(\prod_{s=1}^i a_s^{n_s(2\alpha_s-1)+(\alpha_s-1)} \prod_{u=1}^i c_u y \right). \quad (25)$$

We outline these arguments again below, now adapted to the setting of heterogeneous market parameters. Firstly, the statement obviously holds for $i = 1$ by Lemma 2. In the general inductive step, we show that if \tilde{I}_k is given by (25) with its corresponding $(\alpha_1, \dots, \alpha_k)$, then the statement is also true for \tilde{I}_{k+1} . Noticing that Φ_k and Ψ_k defined for solving the single-period inverse problem with \tilde{I}_k given by (25) must be expressed by (10), we apply Lemma 2 to obtain

that \tilde{I}_{k+1} is unique and given by

$$\tilde{I}_{k+1}(y) = \frac{\prod_{v=1}^{k+1} (1 + b_v)}{b_{k+1} \prod_{j=1}^k b_j^{\alpha_j}} \sum_{n_1=0, \dots, n_{k+1}=0}^{\infty} \left((-1)^{p(n_1, \dots, n_{k+1})} b_{k+1}^{-n_{k+1}} \prod_{t=1}^k b_t^{n_t(1-2\alpha_t)} \right. \\ \left. \times I_0(a_{k+1}^{n_{k+1}} \prod_{s=1}^k a_s^{n_s(2\alpha_s-1)+(\alpha_s-1)} \prod_{u=1}^{k+1} c_u y) \right),$$

if (Φ_k, Ψ_k) satisfies (C1), and given by

$$\tilde{I}_{k+1}(y) = \frac{\prod_{v=1}^{k+1} (1 + b_v)}{\prod_{j=1}^k b_j^{\alpha_j}} \sum_{n_1=0, \dots, n_{k+1}=0}^{\infty} \left((-1)^{p(n_1, \dots, n_{k+1})} b_{k+1}^{n_{k+1}} \prod_{t=1}^k b_t^{n_t(1-2\alpha_t)} \right. \\ \left. \times I_0(a_{k+1}^{-(n_{k+1}+1)} \prod_{s=1}^k a_s^{n_s(2\alpha_s-1)+(\alpha_s-1)} \prod_{u=1}^{k+1} c_u y) \right),$$

if (Φ_k, Ψ_k) satisfies (C2). Therefore, $\tilde{I}_{k+1}(y)$ can be expressed by (25) with the sequence given by $(\alpha_1, \dots, \alpha_k, 1)$ when (C1) is satisfied or $(\alpha_1, \dots, \alpha_k, 0)$ when (C2) is satisfied. This proves the claim and thus shows that I_m is uniquely given by (7). The expression for U_m and X_m^* can be derived similarly as in the proof of Theorem 1. \square

F Proof of Corollary 2

The measurability of the mapping $\mathcal{M} \rightarrow \mathbb{R}$ defined by $(p, u, d) \mapsto U_m(x)$ in the heterogeneous setting of Corollary 2 can be proved analogously as in Corollary 1. \square

G Proof of Proposition 3

We only discuss the case $\theta > 1$. Similar arguments apply when $0 < \theta < 1$.

Let $(D_i)_{i=0, \dots, T}$ and $m \in \mathbb{N}$ be fixed. We first compute the expected performance at times

before the first interaction with the robo-advisor. By Theorem 1 and Theorem 2, we have

$$\begin{aligned}
& \mathbb{E}_1 \left[U_i^{(1)}(X_i^{(m)}) | \mathcal{F}_{i-1} \right] \\
&= D_i p U_i^{(1)} \left(I_i^{(m)} \left(\frac{q}{p} U_{i-1}^{(m)'}(X_{i-1}^{(m)}) \right) \right) + (1 - D_i p) U_i^{(1)} \left(I_i^{(m)} \left(\frac{1-q}{1-p} U_{i-1}^{(m)'}(X_{i-1}^{(m)}) \right) \right) \\
&= D_i p \prod_{j=1}^i \delta_{D_j}^{\frac{1}{\theta}} U_0 \left(I_i^{(m)} \left(\frac{q}{p} U_{i-1}^{(m)'}(X_{i-1}^{(m)}) \right) \right) \\
&\quad + (1 - D_i p) \prod_{j=1}^i \delta_{D_j}^{\frac{1}{\theta}} U_0 \left(I_i^{(m)} \left(\frac{1-q}{1-p} U_{i-1}^{(m)'}(X_{i-1}^{(m)}) \right) \right) \\
&= D_i p \prod_{j=1}^i \delta_{D_j}^{\frac{1}{\theta}} \left(1 - \frac{1}{\theta} \right)^{-1} \delta^{1-\frac{1}{\theta}} q^{1-\theta} p^{\theta-1} \left(X_{i-1}^{(m)} \right)^{1-\frac{1}{\theta}} \\
&\quad + (1 - D_i p) \prod_{j=1}^i \delta_{D_j}^{\frac{1}{\theta}} \left(1 - \frac{1}{\theta} \right)^{-1} \delta^{1-\frac{1}{\theta}} (1-q)^{1-\theta} (1-p)^{\theta-1} \left(X_{i-1}^{(m)} \right)^{1-\frac{1}{\theta}} \\
&= \delta_{D_i}^{\frac{1}{\theta}} \delta^{1-\frac{1}{\theta}} U_{i-1}^{(1)}(X_{i-1}^{(m)}) \left(D_i p^{\theta} q^{1-\theta} + \left(\frac{1-D_i p}{1-p} \right) (1-p)^{\theta} (1-q)^{1-\theta} \right),
\end{aligned}$$

for $i = 1, \dots, m$, where $\delta_{D_j} = \frac{1+b}{c_j^{\theta}(a_j^{-\theta}+b)} = \frac{1}{(D_j p)^{\theta} q^{1-\theta} + (1-D_j p)^{\theta} (1-q)^{1-\theta}}$, $j = 1, 2, \dots, i$, and $\delta = \delta_{D_1}$. Indeed, $U_{i-1}^{(1)}(X_{i-1}^{(m)}) = \prod_{j=1}^{i-1} \delta_{D_j}^{1-\frac{1}{\theta}} U_0(X_{i-1}^{(m)}) = (1 - \frac{1}{\theta})^{-1} \prod_{j=1}^{i-1} \delta_{D_j}^{1-\frac{1}{\theta}} (X_{i-1}^{(m)})^{1-\frac{1}{\theta}}$.

Let $C_1 = p^{\theta} q^{1-\theta}$, $C_2 = (1-p)^{\theta} (1-q)^{1-\theta}$, and consider a new variable $t_i = (\frac{D_i(1-p)}{1-D_i p})^{\theta}$. Clearly, t_i is strictly positive and increasing in D_i . Then $\mathbb{E}_1[U_i^{(1)}(X_i^{(m)}) | \mathcal{F}_{i-1}]$ can be represented by

$$\mathbb{E}_1[U_i^{(1)}(X_i^{(m)}) | \mathcal{F}_{i-1}] = \left(\frac{C_1}{(C_1 + t_i^{-1} C_2)^{\frac{1}{\theta}}} + \frac{C_2}{(t_i C_1 + C_2)^{\frac{1}{\theta}}} \right) U_{i-1}^{(1)}(X_{i-1}^{(m)}) \delta^{1-\frac{1}{\theta}}.$$

Let $f(t_i) = \frac{C_1}{(C_1 + t_i^{-1} C_2)^{\frac{1}{\theta}}} + \frac{C_2}{(t_i C_1 + C_2)^{\frac{1}{\theta}}}$. After taking derivative we have

$$\begin{aligned}
f'(t_i) &= \frac{1}{\theta} C_1 C_2 t_i^{-2} (C_1 + t_i^{-1} C_2)^{-\frac{1}{\theta}-1} - \frac{1}{\theta} C_1 C_2 (t_i C_1 + C_2)^{-\frac{1}{\theta}-1} \\
&= \frac{1}{\theta} C_1 C_2 (t_i C_1 + C_2)^{-\frac{1}{\theta}-1} (t_i^{\frac{1}{\theta}-1} - 1).
\end{aligned}$$

When $\theta > 1$, $U_0(x) > 0$, $\frac{1}{\theta} - 1 < 0$, $f(t_i)$ is increasing first and attains its maximum at $t_i = 1$, which corresponds to $D_i = 1$, and then begins to decrease. Since $f(1) = \frac{C_1 + C_2}{(C_1 + C_2)^{\frac{1}{\theta}}} = \delta^{\frac{1}{\theta}-1}$, we have $\mathbb{E}_1[U_i^{(1)}(X_i^{(m)}) | \mathcal{F}_{i-1}] = U_{i-1}^{(1)}(X_{i-1}^{(m)})$ when $D_i = 1$. Therefore, let $t_{i,\max} = (\frac{D_{i,u}(1-p)}{1-D_{i,u}p})^{\theta}$,

$t_{i,\min} = (\frac{D_{i,d}(1-p)}{1-D_{i,d}p})^\theta$, $f_{D_{i,u}} = \frac{C_1}{(C_1+t_{i,\max}^{-1}C_2)^{\frac{1}{\theta}}} + \frac{C_2}{(t_{i,\max}C_1+C_2)^{\frac{1}{\theta}}}$ and $f_{D_{i,d}} = \frac{C_1}{(C_1+t_{i,\min}^{-1}C_2)^{\frac{1}{\theta}}} + \frac{C_2}{(t_{i,\min}C_1+C_2)^{\frac{1}{\theta}}}$. The value range of $f(t_i)$ is $[\min\{f_{D_{i,u}}, f_{D_{i,d}}\}, \delta^{\frac{1}{\theta}-1}]$, and $\mathbb{E}_1[U_i^{(1)}(X_i^{(m)})|\mathcal{F}_{i-1}]$ is thus bounded between $[\delta^{1-\frac{1}{\theta}}\min\{f_{D_{i,u}}, f_{D_{i,d}}\}U_{i-1}^{(1)}(X_{i-1}^{(m)}), U_{i-1}^{(1)}(X_{i-1}^{(m)})]$ for any possible value of D_i in the interval $[D_{i,d}, D_{i,u}]$.

Let $f_j = \min\{f_{D_{j,u}}, f_{D_{j,d}}\}$ for $j = 2, 3, \dots, m$. According to the above we have

$$\mathbb{E}[U_m^{(1)}(X_m^{(m)})] \leq \mathbb{E}[U_{m-1}^{(1)}(X_{m-1}^{(m)})] \leq \dots \leq U_0(x)$$

and

$$\mathbb{E}[U_m^{(1)}(X_m^{(m)})] \geq \mathbb{E}[\delta^{1-\frac{1}{\theta}}f_m U_{m-1}^{(1)}(X_{m-1}^{(m)})] \geq \dots \geq \prod_{j=1}^m f_j \delta^{m(1-\frac{1}{\theta})} U_0(x).$$

When $T > m$ then, according to our assumptions, the agent interacts at time m with the robo-advisor to update p_m back to the original p and her wealth is reduced from X_m^m to αX_m^m . Since $U_i^{(1)}(\alpha x) = \alpha^{1-\frac{1}{\theta}} U_i^{(1)}(x)$, which one can show similarly as in Example 1, we can repeat the above steps and obtain that

$$\begin{aligned} \mathbb{E}[U_{2m}^{(1)}(X_{2m}^{(m)})] &\in \left[\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^{2m} f_j \delta^{2m(1-\frac{1}{\theta})} U_0(x), \alpha^{1-\frac{1}{\theta}} U_0(x) \right] \\ &= \left[\alpha^{1-\frac{1}{\theta}} \left(\prod_{j=1}^m f_j \right)^2 \delta^{2m(1-\frac{1}{\theta})} U_0(x), \alpha^{1-\frac{1}{\theta}} U_0(x) \right]. \end{aligned}$$

The last equality holds because the choice of intervals are periodic. Repeating the above argument then immediately proves the claim.

To show that f_i is non-increasing in $D_{i,u}$ and non-decreasing in $D_{i,d}$, we notice that $D_{i,u} \geq 1$ and thus $t_{i,\max} \geq 1$ and that $t_{i,\max}$ is increasing in $D_{i,u}$. Therefore, due to the fact that $f(t)$ is decreasing when $t \geq 1$, $f_{D_{i,u}} = f(t_{i,\max})$ is decreasing in $D_{i,u}$. One can show analogously that $f_{D_{i,d}} = f(t_{i,\min})$ is increasing in $D_{i,d}$. Because $f_{D_{i,u}}$ does not depend on $D_{i,d}$ and $f_{D_{i,d}}$ does not depend on $D_{i,u}$, we conclude that $f_i = \min\{f_{D_{i,u}}, f_{D_{i,d}}\}$ is non-decreasing in $D_{i,u}$ and

non-increasing in $D_{i,d}$. □

H Proof of Proposition 4

We only show the proof for $\theta > 1$, similar arguments hold for $0 < \theta < 1$, but note that then $U_0(x)$ is negative.

Let terminal time $T \in \mathbb{N}$ be given, its minimal expected performance for any interaction schedule m is given by

$$\alpha^{(\frac{T}{m}-1)(1-\frac{1}{\theta})} \left(\prod_{j=1}^m f_j \right)^{\frac{T}{m}} \delta^{T(1-\frac{1}{\theta})} U_0(x) = \frac{\left(\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^m (f_j \delta^{1-\frac{1}{\theta}}) \right)^{\frac{T}{m}}}{\alpha^{1-\frac{1}{\theta}}} U_0(x)$$

Apparently, maximising the minimal expected performance or $\left(\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^m (f_j \delta^{1-\frac{1}{\theta}}) \right)^{\frac{T}{m}}$ over the divisor m of T is equivalent to maximising $\left(\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^m (f_j \delta^{1-\frac{1}{\theta}}) \right)^{\frac{1}{m}}$ which is positive for any interaction schedule m , and we denote this optimal schedule by m^* , by considering a large enough T , it can be ensured that $\left(\alpha^{1-\frac{1}{\theta}} \prod_{j=1}^m (f_j \delta^{1-\frac{1}{\theta}}) \right)^{\frac{1}{m}}$ will not continue increasing after T if $m^* = T$. The assertion for the independence of T is thus shown, i.e., any setup with horizon T which is a multiple of m^* must share the same optimal interaction schedule. □

References

- Humoud Alsabah, Agostino Capponi, Octavio Ruiz Lacedelli, and Matt Stern. Robo-advising: Learning investors' risk preferences via portfolio choices. *Journal of Financial Econometrics*, 19(2):369–392, 2021.
- Bahman Angoshtari, Thaleia Zariphopoulou, and Xun Yu Zhou. Predictable forward performance processes: The binomial case. *SIAM Journal on Control and Optimization*, 58(1): 327–347, 2020.

- Mikhail Bekeov, Kevin Lehmann, and Manuel Wittke. Robo advisors: quantitative methods inside the robots. *Journal of Asset Management*, 19(6):363–370, 2018.
- Francois Berrier, Leonard CG Rogers, and Michael Tehranchi. A characterization of forward utility functions. *Statistical Laboratory, University of Cambridge, Cambridge, UK*, 2009.
- Fischer Black and Robert Litterman. Asset allocation: Combining investor views with market equilibrium. *Journal of Fixed Income*, 1(2):7–18, 1991.
- Fischer Black and Robert Litterman. Global portfolio optimization. *Financial Analysts Journal*, 48(5):28–43, 1992.
- Agostino Capponi, Sveinn Olafsson, and Thaleia Zariphopoulou. Personalized robo-advising: Enhancing investment through client interaction. *Management Science*, 2021.
- Wing Fung Chong. Pricing and hedging equity-linked life insurance contracts beyond the classical paradigm: The principle of equivalent forward preferences. *Insurance: Mathematics and Economics*, 88:93–107, 2019.
- John C Cox, Stephen A Ross, and Mark Rubinstein. Option pricing: A simplified approach. *Journal of Financial Economics*, 7(3):229–263, 1979.
- Xiangyu Cui, Duan Li, Xiao Qiao, and Moris S Strub. Risk and potential: An asset allocation framework with applications to robo-advising. *Available at SSRN 3302111*, 2019.
- Min Dai, Hanqing Jin, Steven Kou, and Yuhong Xu. A dynamic mean-variance analysis for log returns. *Management Science*, 67(2):1093–1108, 2021a.
- Min Dai, Hanqing Jin, Steven Kou, and Yuhong Xu. Robo-advising: A dynamic mean-variance approach. *Digital Finance*, pages 1–17, 2021b.
- Xue Dong He, Moris S Strub, and Thaleia Zariphopoulou. Forward rank-dependent performance criteria: Time-consistent investment under probability distortion. *Mathematical Finance*, 31(2):683–721, 2021.

- Vicky Henderson and David Hobson. Horizon-unbiased utility functions. *Stochastic Processes and their Applications*, 117(11):1621–1641, 2007.
- Ying Hu, Gechun Liang, and Shanjian Tang. Systems of ergodic BSDEs arising in regime switching forward performance processes. *SIAM Journal on Control and Optimization*, 58(4):2503–2534, 2020.
- Sigrid Källblad, Jan Obłój, and Thaleia Zariphopoulou. Dynamically consistent investment under model uncertainty: the robust forward criteria. *Finance and Stochastics*, 22(4):879–918, 2018.
- Sigrid Källblad. Black’s inverse investment problem and forward criteria with consumption. *SIAM Journal on Financial Mathematics*, 11(2):494–525, 2020.
- Dimitri Kramkov and Walter Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, pages 904–950, 1999.
- Rainer Kress, V Maz’ya, and V Kozlov. *Linear integral equations*, volume 82. Springer, 1989.
- Marek Kuczma, Bogdan Choczewski, and Roman Ger. *Iterative Functional Equations*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1990.
- Gechun Liang and Thaleia Zariphopoulou. Representation of homothetic forward performance processes in stochastic factor models via ergodic and infinite horizon BSDE. *SIAM Journal on Financial Mathematics*, 8(1):344–372, 2017.
- Marek Musiela and Thaleia Zariphopoulou. Investments and forward utilities. *Preprint*, 2006.
- Marek Musiela and Thaleia Zariphopoulou. Optimal asset allocation under forward exponential performance criteria. In *Markov processes and related topics: a Festschrift for Thomas G. Kurtz*, pages 285–300. 2008.

- Marek Musiela and Thaleia Zariphopoulou. Portfolio choice under dynamic investment performance criteria. *Quantitative Finance*, 9(2):161–170, 2009.
- Marek Musiela and Thaleia Zariphopoulou. Portfolio choice under space-time monotone performance criteria. *SIAM Journal on Financial Mathematics*, 1(1):326–365, 2010.
- Sergey Nadtochiy and Michael Tehranchi. Optimal investment for all time horizons and Martin boundary of space-time diffusions. *Mathematical Finance*, 27(2):438–470, 2017.
- Andrei D Polyanin and Alexander V Manzhirov. *Handbook of integral equations*. Chapman and Hall/CRC, 2008.
- Mykhaylo Shkolnikov, Ronnie Sircar, and Thaleia Zariphopoulou. Asymptotic analysis of forward performance processes in incomplete markets and their ill-posed HJB equations. *SIAM Journal on Financial Mathematics*, 7(1):588–618, 2016.
- Moris S Strub and Xun Yu Zhou. Evolution of the Arrow–Pratt measure of risk-tolerance for predictable forward utility processes. *Finance and Stochastics*, 25(2):331–358, 2021.
- Stephen M Zemyan. *The classical theory of integral equations: a concise treatment*. Springer Science & Business Media, 2012.
- Gordan Žitković. A dual characterization of self-generation and exponential forward performances. *The Annals of Applied Probability*, 19(6):2176–2210, 2009.