# Bet on a bubble asset ? An optimal portfolio allocation strategy

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**Abstract**

We discuss portfolio allocation when one asset exhibits phases of locally explosive behavior. We model the conditional distribution of such an asset through mixed causal-non-causal models which mimic well the speculative bubble behaviour. Relying on a Taylor-series-expansion of a CRRA utility function approach, the optimal portfolio(s) is(are) located on the mean-variance-skewness-kurtosis efficient surface. We analytically derive these four conditional moments and show in a Monte-Carlo simulations exercise that incorporating them into a two-assets portfolio optimization problem leads to substantial improvement in the asset allocation strategy. All performance evaluation metrics support the higher out-of-sample performance of our investment strategies over standard benchmarks such as the mean-variance and equally-weighted portfolio. An empirical illustration using the Brent oil price as the speculative asset confirms these findings.

*Keywords:* non-causal process, $\alpha$-stable, asset allocation, utility function, out-of-sample performance
*JEL:* C51, C22, G12

## 1. Introduction

A close look at the dynamics of various asset prices, that are sometimes called speculative assets, reveals the presence of phases of locally explosive behaviors, i.e. increasing patterns followed by a burst. Called rational asset pricing bubbles when due to rational deviations from the fundamental value [see Blanchard and Watson, 1982, Tirole, 1985], these phenomena have been detected more and more accurately in the financial markets across the world together with the more traditional properties of heavy-tailed marginal distributions and volatility clustering.

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A rich theoretical literature has been focusing on two aspects of this phenomenon: the investment problem and the financial economic implications, [see e.g. Davis and Lleo, 2013]. To explain how a bubble originates in the market, researchers generally relied on standard martingale theory of bubbles [Biagini et al., 2014, Jarrow et al., 2010, Protter, 2012], or added further assumptions such as portfolio constraints or defaultable claims, [see Biagini and Nedelcu, 2015, Hugonnier, 2012, Jarrow et al., 2012]. The impact of bubbles on economic growth [Martin and Ventura, 2012, Carvalho et al., 2012] or on unemployment [Hashimoto and Im, 2016, 2019, Miao et al., 2016] has also been scrutinized recently.

But this phenomenon has not been thoroughly gauged so far through the lens of portfolio allocation, although optimal portfolio selection has been a major topic in finance since the works of Markowitz [1952]. The scarcity of this literature may be explained by the distributional specificities of bubble asset prices and the risk they incur although, from a financial perspective, investors are certainly interested in constructing portfolios hedging bubble burst risk. Indeed, traditional portfolio theory is consistent with expected utility and its von Neumann-Morgenstern axioms of choice when either asset returns are normally distributed (i.e., higher moments are irrelevant), or investors have a quadratic utility function [see e.g. Samuelson, 1967]. But these assumption were shown not to be empirically justified [see Mandelbrot, 1963, Ang et al., 2006, Massacci, 2017, Ingersoll, 1975, Scott and Horvath, 1980, among others].

This lead researchers and practitioners to intensively work on new portfolio allocation strategies, which, among others, pay attention to higher order moments, namely asymmetry and fat-tailness [see Briec et al., 2013, Kolm et al., 2014] for literature reviews). A wide variety of approaches have been proposed in this literature: Taylor expansion of the expected utility [Jondeau and Rockinger, 2006, 2012, Guidolin and Timmermann, 2008, Martellini and Ziemann, 2010], Gram-Charlier expansion of downside risk measures [Favre and Galeano, 2002, León and Moreno, 2017, Zoia et al., 2018, Lassance and Vrins, 2021], the shortage function of [Briec et al., 2007, 2013, to name but a few].

A portfolio strategy based on an accurate characterization of the mechanism generating financial bubbles seems necessary to avoid misleading outcomes. However, neither ARMA nor (G)ARCH / stochastic volatility models, traditionally used to characterize the predictive distribution of returns, are able to mimic such bubble behaviours. To our knowledge, only the paper by Ghahtarani [2021] discusses portfolio allocation in bubble conditions. The author introduces a new portfolio risk measure and shows that it can perform better than classical risk measures in bubble situations. To this aim, he uses a fuzzy neural network model to compute scenario paths of end-horizon market value. But the approach is not specifically designed for portfolio allocation, it operates in multiple steps and requires accounting for uncertainty surrounding the fundamental and market value predictions.

We contribute to this literature by documenting the attractiveness of portfolio strategies that account explicitly for the distributional characteristics of bubble assets. More precisely, we exploit very recent theoretical results on non-causal models to appropriately characterize the conditional distribution of asset prices exhibiting bubble behaviour. Indeed, non-causal autoregressive processes with stable distributed errors appear to be fit to model speculative financial bubbles as they mimic well locally explosive patterns [see e.g. Gourieroux and Zakoian, 2017].

Our approach is anchored in the classical theoretical rational-expectations bubble framework proposed by Blanchard and Watson [1982]. A bubble occurs when prices temporarily deviate from the fundamental value. But if Blanchard and Watson's model features successive bubble/burst cycles, the non-causal model may generate more realistic price dynamics where bubble events intersperse calmer periods. Besides, the gradual collapse in the dynamics of mixed causal-noncausal model (hereafter MAR) reconciles the rational expectations bubbles with regular variation tail indexes above 1, a well-documented statistical property of financial data, [see Lux and Sornette, 2002].[5] Most importantly, they exhibit surprising features such as a predictive distribution with lighter tails than the marginal distribution, which allows one to obtain predictions of higher-moments that are expected to be of crucial importance for the (non-)investment decision. Indeed, this framework relaxes the finite variance constraint while insuring the stationarity of the process, [see Gourieroux et al., 2020, on the existence of multiple stationary nonlinear equilibria in bubble models].

By relying on the results of Fries [2021], we derive the first four conditional moments of an $\alpha$-stable MAR(1,1) process and show that incorporating them into a two-assets portfolio optimization problem can lead to substantial improvement in the asset allocation strategy. For this, we consider the standard Taylor-series-expansion of a CRRA utility function approach à la Jondeau

[5]We defer the reader to Fries [2021] for further discussion on the link between non-causal models and rational bubbles à la Blanchard and Watson [1982].

and Rockinger [2006, 2012], [see also Martellini and Ziemann, 2010]. The optimal portfolio(s) is(are) located on the mean-variance-skewness-kurtosis efficient surface in the sense that no other portfolio can dominate it on all four moments. But since there is evidence that standard utility functions are locally quadratic and higher-order moments may not significantly impact portfolio selection [see e.g. Markowitz, 2014], we also consider, as a robustness check, a polynomial-goal-programming (PGP) problem so as to find a portfolio on the higher-moment efficient surface without the need to specify a utility function. The economic value of our strategy is compared with standard benchmarks such as the mean-variance and equally-weighted portfolios.

In contrast to Ghahtarani [2021], if his machine learning framework were to be used for portfolio allocation, our approach is not scenario-depedent. Besides, the non-causal framework also presents the advantage of ease of interpretability, in the sense that the solution(s) of the portfolio allocation problem can be traced back to the conditioning value of the bubble asset dynamics and the higher-order moments of the conditional return distribution.

A set of Monte-Carlo simulations emphasizes the reliability of our approach. Indeed, the portfolio strategies obtained when estimating the MAR(1,1) parameters are clustered around the theoretical optimal ones, i.e. based on the true parameters, and their dispersion reduces quickly as the sample size increases. This indicates that estimation uncertainty does not affect much the portfolio allocation problem. Note, however, that the starting value of the speculative asset $X_t = x$ matters a lot in the selection of the optimal investment share and horizon, which is not the case of the no-bubble asset. This is expected to lead to investment strategies that outperform standard benchmarks such as the mean-variance and equally-weighted portfolios. We study this intuition by simulation and rely on several performance evaluation measures to gauge the out-of-sample relative performance of our approach. All methods used, i.e. terminal wealth, opportunity cost, Sharpe and modified Sharpe ratios, support the superiority of our portfolio allocation strategy. The difference is particularly significant when the conditioning values $X_t = x$ are in the tails of the marginal distribution of the process, i.e. when the first asset is indeed close to the peak of a bubble period, which is of outmost importance for the investor.

An empirical illustration using the Brent oil price as the speculative asset confirms these simulation-based results. As a preliminary step to select candidate assets, we test for the presence of bubbles in asset price dynamics by relying on the recent generalized-sup ADF test of Phillips et al. [2015] that is appropriate for rational bubble frameworks, among others.[6] The pseudo-out-of-sample performance of our portfolio allocation approach is then compared to that of the two benchmark models. The test by Ardia and Boudt [2015] is particularly useful to statistically assess the significance of the difference between the modified Sharpe ratios. All findings support the superiority of our approach.

The paper is structured as follows. In Section 2 we introduce the proposed allocation problem. Section 3 introduces a Monte Carlo study that discusses the impact of parameter uncertainty on the optimal strategies. In Section 4 we conduct a simulation-based out-of-sample horce-race with standard benchmarks to evaluate the relative economic value of our approach, while Section 5 details the empirical illustration.. Finally, Section 6 concludes and the Appendices include proofs of results and additional results.

## 2. Bubble-riding allocation problem

Hedging bubble asset risk is nowadays a particularly important issue for an investor handling speculative assets. In this section, we provide a unified framework to solve the allocation problem in presence of an asset exhibiting a bubble behaviour. First, we formally introduce the portfolio allocation problem and then provide the necessary quantities to compute the conditional moments of portfolio return distribution when the speculative asset price is modeled as a mixed causal-noncausal process. Finally, we briefly review the methods that will be used to evaluate the economic value of the optimal portfolio strategies.

### *2.1. Optimal portfolio allocation*

We investigate the asset allocation problem in the context with a speculative asset price $X_t$, for which the dynamics of higher order conditional moments is of particular importance, and a safer one, $S_t$. Two approaches that account for higher-order moments in the choice of the optimal portfolio have gained investors' attention to date and are considered in our analysis. The first is based on a Taylor expansion of the expected utility function, while the latter consists in the Polynomial Goal Programming (PGP) model. We privilege the CRRA utility function because it

[6] Early tests for rational bubbles relied on Shiller [1981]'s variance bounds test, West [1987, 1988]'s two step procedure or cointegration tests, but these approaches are subjected to multiple issues.

is probably economically the most relevant preference family, as it realistically assumes that risk aversion is relatively constant over wealth levels, [see also Jondeau and Rockinger, 2006, 2012, and references therein]. A complementary analysis, based on the PGP model, is available in Appendix B [see De Athayde and Flôres Jr, 2004, for arguments in favour of this approach].

We consider an investor endowed with wealth $W_t$ at present date $t$, who allocates her portfolio constituted of these two assets to maximize the expected utility $U(W)$ over her end-of-period wealth $W_{t+H}$. The initial wealth is innocuous to the optimization problem and arbitrarily set to one. The investor has an investment horizon $H$: at date $t$, she will decide of the share $\omega$ (resp. $1-\omega$) to invest in the speculative asset (resp. safer asset), and of the intermediate horizon $h \leq H$ at which she commits to liquidate its holding of speculative asset and to invest the proceedings in the safer asset until $t+H$. Short selling is allowed, hence portfolio weights can take both positive and negative values. This leads to an optimization problem of the terminal wealth $W_{t+H}$ or, equivalently, of the overall return $R_{t+H} = (W_{t+H} - W_t)/W_t$ in both the allocation $\omega$ and the intermediate horizon $h$.

We assume that the speculative asset's price $X_t$ follows a mixed causal-noncausal stable AR process, i.e. MAR(1,1), with a non-zero location parameter. This choice is motivated by the recent econometric literature that proved non-causal models to be a convenient way to model locally explosive phenomena such as speculative bubbles, while featuring heavy-tailed marginals and conditional heteroscedastic effects generally encountered in financial data [see e.g. Cavaliere et al., 2020, Fries and Zakoian, 2019, Gourieroux and Jasiak, 2018, Gourieroux and Zakoian, 2017]. The safer asset is assumed to follow a geometric Brownian motion dynamics with drift $v$ and volatility $\varsigma$. The price processes $(X_t)$ and $(S_t)$ will be assumed independent, which provides a nice framework for hedging purposes.[7]

For a given strategy $(\omega, h)$, the terminal wealth can be expressed as

$$W_{t+H} = \frac{S_{t+H}}{S_{t+h}} \Big( \omega \frac{X_{t+h}}{X_t} + (1-\omega) \frac{S_{t+h}}{S_t} \Big),$$

[7] In this framework, the independence hypothesis does not appear as a strong assumption. As we focus on investment during periods where an asset price exhibits a bubble behaviour, its dynamics cannot be correlated over this time-interval with that of a safe(r) asset. In practice it is reasonable to think of the second *asset* as a well-diversified portfolio whose constituents do not exhibit any bubble behaviour. This makes the second asset an attractive hedge against the risk of bubble collapse in the first asset.

or alternatively, in terms of returns,

$$W_{t+H} = 1 + R_{t+H},$$

where the terminal portfolio return $R_{t+H}$ writes

$$R_{t+H} = \left(1 + r^S_{t+h,t+H}\right)\left(\omega r^X_{t,t+h} + (1-\omega) r^S_{t,t+h} + 1\right) - 1,$$

with $r^S_{t+h,t+H} := S_{t+H}/S_{t+h} - 1$, $r^S_{t,t+h} := S_{t+h}/S_t - 1$, and $r^X_{t,t+h} := X_{t+h}/X_t - 1$, the asset's returns in-between the key investment events.

In this framework, we follow Jondeau and Rockinger [2006, 2012] to approximate the allocation problem.[8] The CRRA utility maximization program of the fourth order Taylor approximation around the expected terminal wealth is

$$\max_{(\omega,h)} \mathbb{E}[U(W_{t+H}|X_t, S_t)] \approx \sum_{k=0}^{4} \frac{U^{(k)}(\overline{W}_{t+H})}{k!} \mathbb{E}\left[(W_{t+H} - \overline{W}_{t+H})^k | X_t, S_t\right], \tag{1}$$

with $U(c) = c^{1-\gamma}/(1-\gamma)$ for a risk aversion parameter $\gamma > 0$ and $\overline{W}_{t+H} = \mathbb{E}\left[W_{t+H}|X_t, S_t\right]$. The investor's preference (or aversion) toward the $k^{\text{th}}$ moment is directly given by the $k^{\text{th}}$ derivative of the utility function. The effects of the third and fourth moments on the approximated expected utility are positive and negative, respectively, and correspond to financial theory [see Scott and Horvath, 1980]. The expected utility also depends on the central conditional moments of the distribution of terminal wealth, which can be expressed in terms of conditional moments of the portfolio return distribution as $\mathbb{E}\left[(W_{t+H} - \overline{W}_{t+H})^k | X_t, S_t\right] = \mathbb{E}\left[(R_{t+H} - \overline{R}_{t+H})^k | X_t, S_t\right]$, since $\overline{W}_{t+H} = 1 + \overline{R}_{t+H}$ with $\overline{R}_{t+H} := \mathbb{E}[R_{t+H}|X_t, S_t]$. It is just a matter of algebra using the independence between $(X_t)$ and $(S_t)$ to express the objective functions in terms of the conditional moments of the speculative asset price, $\mathbb{E}\left[X^p_{t+h}|X_t\right]$, $p = 1, 2, 3, 4$, that are detailed in the next subsection, and the parameters (see Appendix A.1 for further computational details).

[8]Lhabitant et al. [1998] has shown that the infinite Taylor series expansion converges to the expected utility in the CRRA case for wealth levels between 0 and $2\overline{W}$ that appear to be large enough for stocks and bonds regardless of the degree of non-normality, in particular when short-selling is prohibited.

*2.2. Conditional moments of MAR(1,1) $\alpha$-stable processes*

In this subsection we discuss the existence and derivation of the first four conditional moments of the speculative asset price. As the econometric literature has identified MAR processes to be appropriate for financial bubble modelling, we rely on them, [see e.g. Hecq and Voisin, 2021, and references therein].[9]

Let $(X_t)$ be the $\alpha$-stable solution of the MAR(1,1) process $X_t = \varphi^\circ X_{t+1} + \varphi^\bullet X_{t-1} + \varepsilon_t$, with i.i.d. $\alpha$-stable errors, $\varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(\alpha, \beta, \sigma, \mu)$ and $\alpha \neq 1$ (for simplicity), $\beta \in [-1, 1]$, and $\sigma > 0$. The process is well defined and strictly stationary for $|\varphi^\circ| < 1$, $|\varphi^\bullet| < 1$, and $\varphi^\circ \neq \varphi^\bullet$. It then has a $MA(\infty)$ representation $X_t = \sum_{k\in\mathbb{Z}} a_k \varepsilon_{t+k}$, whose coefficients satisfy $\sum\limits_{k\in\mathbb{Z}} |a_k|^s < +\infty$ for some $s \in (0, \alpha) \cap [0, 1]$. Without loss of generality, in the following we assume that the shift $\mu$ is null, but in practice we handle the possibility of $\mu \neq 0$ by relying on a simple transformation of the conditional moments obtained with zero location parameter to those associated with a non-null shift [see Section 2 in Fries, 2021].

Now let $\mathbf{X}_t = (X_t, X_{t+h})$ denote the bivariate stable vector obtained from $X_t$ for horizon $h \geq 1$. Proposition 3.1 *i*) in Fries [2021] then applies and states the condition of existence of higher-order conditional power moments, although the marginal variance of the process $X_t$ is infinite. In particular, the conditional moments up to integer order $p$, $\mathbb{E}[|X_{t+h}|^p|X_t]$, may exist as long as $\nu \geq 0$ exists such that $\sum\limits_{k\in\mathbb{Z}} (a_k^2 + a_{k-h}^2)^{\frac{\alpha+\nu}{2}} |a_k|^{-\nu} < +\infty$ and $0 \leq p < \min(\alpha + \nu, 2\alpha + 1)$, where $(a_k,\, a_{k-h})$ are the coefficients of the infinite moving average representation of the process. The more *anticipative, i.e. noncausal* the process, the larger $\nu \geq 0$, which insures the existence of all conditional moments up to order $2\alpha + 1$ at all prediction horizons when $(a_k)$ decays geometrically or hyperbolically for e.g..

**Proposition 1.** *For $\alpha \neq 1$, the moments $\mathbb{E}[X_{t+h}^p|X_t]$, $p \leq 4$, when they exist, are given by Theorems 2.1 and 2.2 in Fries [2021] as a function of four quantities, $\sigma_1^\alpha$, $\beta_1$, $\kappa_p$, and $\lambda_p$ and a family of functions $\mathcal{H}$.*

[9] More generally, any MARMA model could be used, but this would engender a cost related to the numerical approximation of the $MA(\infty)$ coefficients [see Fries, 2021]. We prefer a more parsimonious approach for which we can obtain the formulas for these coefficients in closed form.

*We demonstrate that in the case of a MAR(1,1) process these constants are equal to*

$$
\begin{aligned}
\sigma_1^\alpha =& \sigma^\alpha \frac{1-|\varphi^\circ\varphi^\bullet|^\alpha}{(1-\varphi^\circ\varphi^\bullet)^\alpha(1-|\varphi^\circ|^\alpha)(1-|\varphi^\bullet|^\alpha)},\\
\beta_1 =& \beta \frac{1-\varphi^{\circ<\alpha>}\varphi^{\bullet<\alpha>}}{1-|\varphi^\circ|^\alpha|\varphi^\bullet|^\alpha}\frac{(1-|\varphi^\circ|^\alpha)(1-|\varphi^\bullet|^\alpha)}{(1-\varphi^{\circ<\alpha>})(1-\varphi^{\bullet<\alpha>})},\\
\kappa_p =& \frac{\varphi^{\bullet hp}(1-|\varphi^\circ|^\alpha)+(\varphi^{\circ -hp}|\varphi^\circ|^{h\alpha})(1-|\varphi^\bullet|^\alpha)}{1-|\varphi^\circ\varphi^\bullet|^\alpha}\\
&+\frac{(\varphi^{\bullet hp}|\varphi^\circ|^\alpha(\varphi^\bullet\varphi^\circ)^{-p}-\varphi^{\circ -hp}|\varphi^\circ|^{\alpha h})(1-|\varphi^\circ|^\alpha)(1-|\varphi^\bullet|^\alpha)}{(1-|\varphi^\circ|^\alpha(\varphi^\bullet\varphi^\circ)^{-p})(1-|\varphi^\circ\varphi^\bullet|^\alpha)},\\
\lambda_p =& \frac{\varphi^{\bullet hp}(1-|\varphi^\circ|^{<\alpha>})+(\varphi^{\circ -hp}\varphi^{\circ<\alpha>h})(1-\varphi^{\bullet<\alpha>})}{1-\varphi^{\circ<\alpha>}\varphi^{\bullet<\alpha>}}\\
&+\frac{(1-\varphi^{\circ<\alpha>})(1-\varphi^{\bullet<\alpha>})}{1-\varphi^{\circ<\alpha>}(\varphi^\bullet\varphi^\circ)^{-p}}\frac{(\varphi^{\bullet hp}\varphi^{\circ<\alpha>}(\varphi^\bullet\varphi^\circ)^{-p}-\varphi^{\circ -hp}\varphi^{\circ<\alpha>h})}{1-\varphi^{\circ<\alpha>}\varphi^{\bullet<\alpha>}},
\end{aligned}
$$

*where* $y^{<\alpha>} = \mathrm{sign}(y)|y|^\alpha$ *for any* $y \in \mathbb{R}$. $\sigma_1$ *and* $\beta_1$ *denote the scale and asymmetry parameters of the marginal distribution of* $X_t$*, whereas the constants* $\kappa_p$ *and* $\lambda_p$*,* $p > 2$*, generalize standard dependence measures invoked in the literature to powers of* $X_t$ *and* $X_{t+h}$ *in the asymmetric case. At the same time,* $\mathcal{H}$ *contains functions related to the marginal density of the stable random variable* $X_t$ *and for* $n \in \mathbb{N}$*,* $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}$*,* $x \in \mathbb{R}$ *is defined as*

$$
\mathcal{H}(n,\theta;x) = \int_0^{+\infty} e^{-\sigma_1^\alpha u^\alpha} u^{n(\alpha-1)}\big(\theta_1\cos(ux-\alpha\beta_1\sigma_1^\alpha u^\alpha)+\theta_2\sin(ux-\alpha\beta_1\sigma_1^\alpha u^\alpha)\big)du.
$$

*Proof.* See Appendix A.2 □

**Remark 1.** *The conditional moments can be easily computed for* $p \leq 4$ *and* $h \geq 1$ *once the functions* $\mathcal{H}(n,\theta;x)$ *are evaluated for* $n = 2,3,4$ *by following the approach discussed by Fries [2021] and originally proposed by Samorodnitsky et al. [1996] for the conditional expectation.*

**Remark 2.** *The asymptotic expressions for the conditional moments with respect to the conditioning variable, i.e. when* $X_t$ *becomes large, given in Proposition 2.1 of Fries [2021] and stated below, remain valid in the MAR(1,1) case when* $\sigma_1^\alpha$*,* $\beta_1$*,* $\kappa_p$*, and* $\lambda_p$ *are replaced by the expressions given in Proposition 1 above. To be more precise, if the conditional moment of order* $p$ *of a bivariate* $\alpha$*-stable vector exists and*

$|\beta_1| \neq 1$, *then,*

$$x^{-p}\mathbb{E}[X_{t+h}^p | X_t = x] \underset{x\to\infty}{\to} \frac{\kappa_p + \lambda_p}{1+\beta_1}, \quad x^{-p}\mathbb{E}[X_{t+h}^p | X_t = x] \underset{x\to-\infty}{\to} \frac{\kappa_p - \lambda_p}{1-\beta_1}. \tag{2}$$

### 2.3. *Performance evaluation measures*

Several investment ratios, e.g. the Sharpe, Sortino, and Omega ratios, and relative performance measures, e.g. the opportunity cost or performance fee (OC) and the Graham–Harvey metric, have been used in the literature to evaluate portfolios' performance, [see e.g. Jondeau and Rockinger, 2006, 2012, González-Pedraz et al., 2015]. But since standard ratios ignore investors' positive preferences for odd moments and aversion to even moments, they are not appropriate for investments with non-normal returns. Several alternatives have been proposed, such as the modified Sharpe ratio (mSharpe) of Favre and Galeano [2002], Gregoriou and Gueyie [2003], which uses as a risk measure an estimator for Value-at-Risk based on the Cornish–Fisher expansion and the first four moments of the return distribution.

In this paper we employ the OC measure to evaluate the out-of-sample performance of our strategy relatively to two traditional benchmarks, the equally-weighted portfolio (EW) and the standard mean-variance (MV) portfolio. This corresponds to the amount that needs to be added to the return of a competing benchmark strategy so that the investor becomes indifferent to the portfolio decision based on our framework. We also report the mSharpe ratio and, for comparison reasons, the Sharpe ratio. In the simulation-based analysis we test the equality of their medians over the out-of-sample for the various competing portfolio strategies, while in the empirical illustration, will rely on the mSharpe ratio comparison test proposed by Ardia and Boudt [2015] to further compare the different allocation strategies in out-of-sample.

## 3. Monte Carlo experiments

As the investor does not have perfect knowledge of the parameters of the distribution of the speculative asset, we investigate the impact of parameter estimation on portfolio allocation in a Monte-Carlo experiment. We adopt a parametric plug-in estimation approach and proceed in two steps.[10] First, we gauge the sensitivity of the conditional moments of returns to parameter estimation and then we look into the variability this induces in the optimal portfolio strategy.

We simulate $M = 2000$ trajectories of $N = \{1000, 2000, 5000\}$ observations from the MAR(1,1) process $(1 - 0.9F)(1 - 0.3B)X_t = \varepsilon_t$ where $\varepsilon_t \overset{i.i.d.}{\sim} S(1.8, 0.5, 0.1, 2)$. We then estimate the conditional power moments by replacing the theoretical constants $\sigma_1^\alpha$, $\beta_1$, $\kappa_p$, $\lambda_p$. in Proposition 1 by their empirical counterparts computed by plugging-in the MAR(1,1) parameter estimates obtained by Maximum Likelihood.[11]

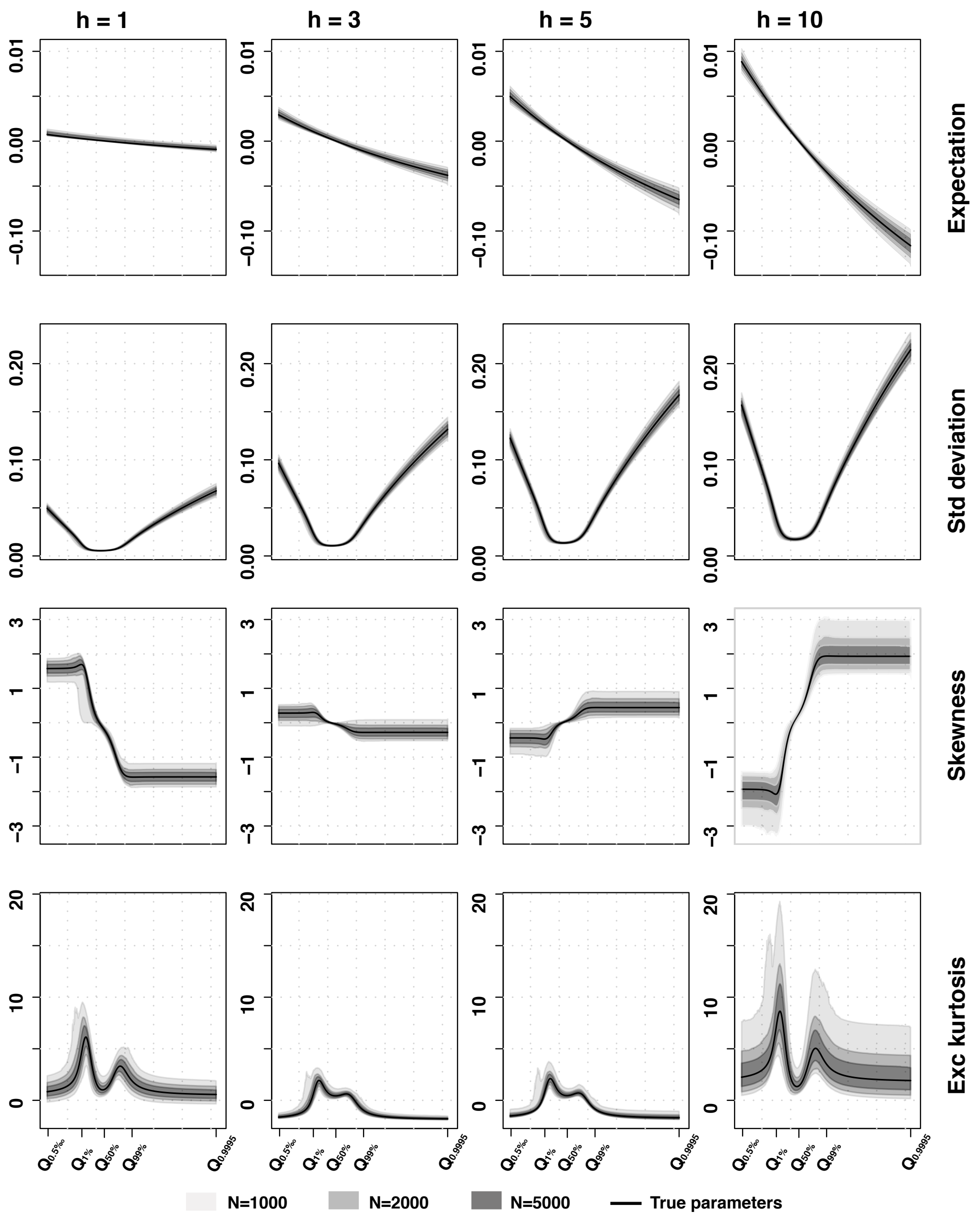
The results are displayed in Figure 1 for prediction horizons $h = 1, 3, 5, 10$ and conditioning values $x \in (24.5 - 36.2)$ that correspond to the 0.05% and 99.95% quantiles of the marginal distribution of $X_t$. They take the form of a pointwise 5% - 95% interquartile interval of the conditional moment estimators for each sample size $N$. Notice that the theoretical conditional moments, based on the true values of the parameters and represented by a black line, always belong to the empirical interquartile range. More precisely, the interquantile intervals are very narrow around most of the true conditional moments curves, even for small sample sizes. They are slightly larger for higher-order moments and large horizons when $N = 1000$ but narrow down fast as the sample size increases. Overall, the plug-in method appears to be a good way to estimate the conditional moments even when the conditioning values $X_t = x$ are in the tails of the marginal distribution of the process.

In the second step we hence investigate the impact of parameter estimation on the selected portfolios. The simulated conditional moments of returns obtained from the ML estimates of the MAR(1,1) process are plugged in the CRRA portfolio optimization program to get the optimal portfolio strategi(es) in the form of couples $(\omega^*, h^*)$, which define the part of the wealth to invest in the bubble asset and the horizon of this investment given that the overall investment horizon is fixed to $H = 250$ periods, i.e., a year of daily trading activity. To be more precise, we search for optima $(\omega^*, h^*)$ in the set $[-1, 1] \times [0, 250]$, thus allowing short strategies. As the optimization program is likely non-convex, several strategies may lead to the same terminal wealth, and in this

[10] A model-free non-parametric approach could also be envisaged, but it would engender a dramatic loss in efficiency, especially for conditioning values $X_t = x$ far away from the central values of the process $(X_t)$, [see Fries, 2021, Supplementary Material].

[11] To facilitate the estimation, we initialize the parameters of the $\alpha$ stable distribution by relying on the approach of McCulloch [1986]. Provided the ML estimator is consistent, which is the case for the one used here [see Andrews et al., 2009], the plug-in estimators of the conditional moments will also be consistent.

Figure 1: Conditional moments of return distribution




*Notes:* First four conditional moments of returns when the price series follows a MAR(1,1) process $(1-0.9F)(1-0.3B)X_t = \varepsilon_t$ with $\varepsilon_t \overset{i.i.d.}{\sim} S(1.8, 0.5, 0.1, 2)$. The conditional moments are obtained for conditioning values $X_t = x \in (24.5, 36.2)$, i.e. 99.9% of the probability mass of the marginal distribution of $X_t$ is supported on this interval. The black line represents the theoretical moments, whereas the gray shaded areas correspond to the simulated conditional moments based on 2000 draws from the above $\alpha$-stable distribution for the three sample sizes. In the latter case the MAR(1,1) parameters are estimated by Maximum Likelihood and plugged in the formulae of Proposition 1. The results are displayed for three horizons, $h = 1, 3, 5, 10$ and three sample sizes, $N = 1000, 2000, 5000$.

case they are all labeled as optimal strategies. We round $\omega^*$ to the closest percentage point and report $h^*$ in weeks. Besides, by convention, if either $\omega^* = 0$ or $h^* = 0$, we report $(\omega^*, h^*) = (0, 0)$. For the safer asset, we set $\upsilon$ and $\varsigma$ so that the annual return and volatility equal 2%.

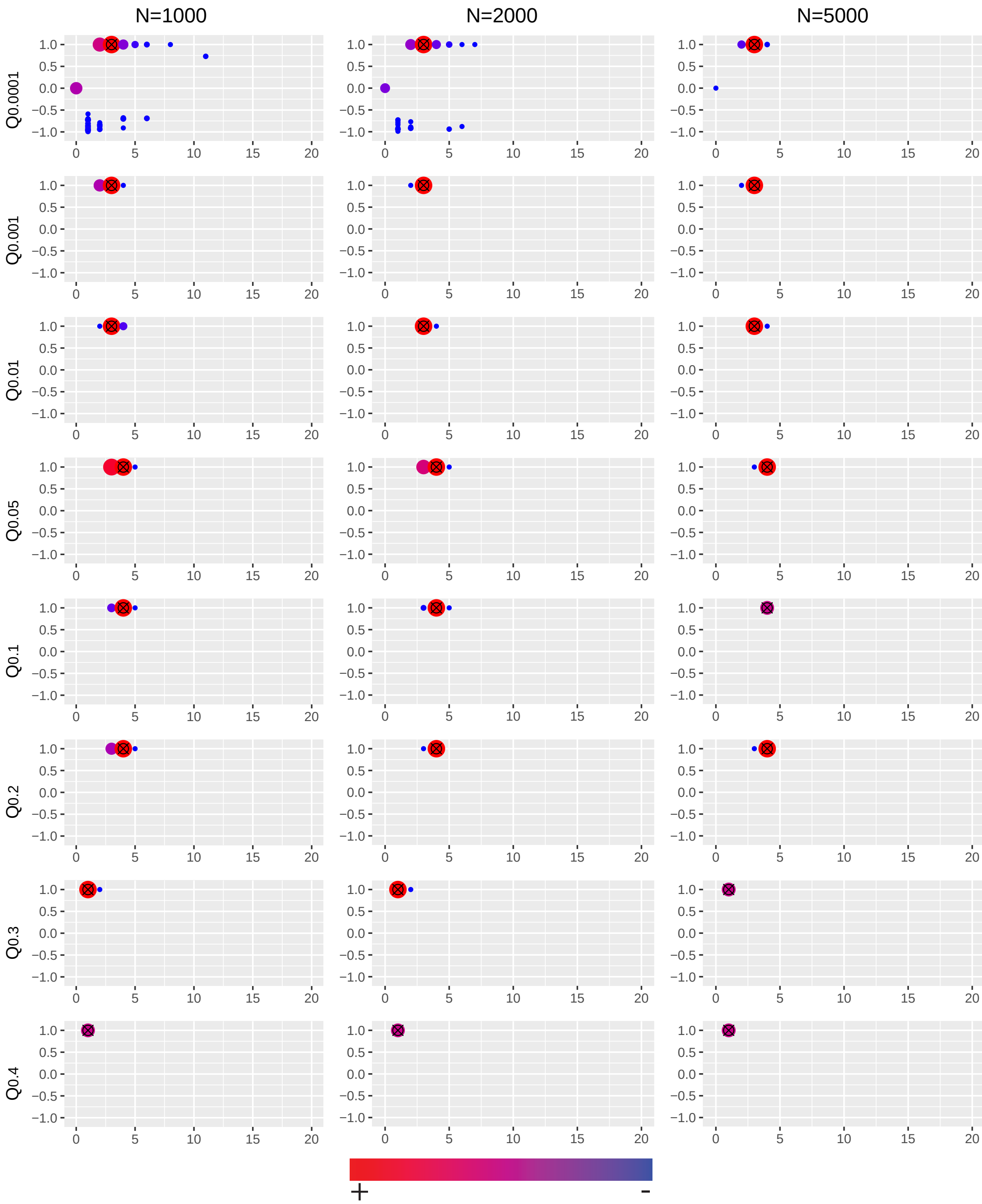
Figures 2 and 3 propose a visualization of the optimal investment strategies if the DGP were known and of the impact of parameter estimation on the selected optimal portfolios for a CRRA investor with risk aversion parameter $\gamma = 10$. For each starting value of the speculative asset $X_t = x$ defined by a specific quantile of its distribution and each sample size $N$, we plot the mass repartition of the estimated strategies across the 2000 simulations in the share-horizon space. The bigger and redder the dots, the larger the mass of portfolios falling in that area. Roughly speaking, a red circle corresponds to more than 1000 identical strategies, a violet one indicate more than 500 identical ones, whereas the smallest blue dots represent between 5 and 50 identical strategies.[12] The optimal strategies under the hypothesis that the investor knows the parameters of the speculative asset dynamics are denoted by black target symbols.

While the starting value of $S_t$ does not matter, the starting value of $X_t$ deeply modifies the investment landscape. The first figure looks into the case of conditioning values in the lower conditional quantiles of $X_t$. The CRRA investor bets on a rising value of the speculative asset and opts for a full investment in it ($\omega^* = 1$) over a short horizon, $h^* < 5$. This long strategy is the only optimal portfolio allocation in this setup, i.e., the equilibrium is unique. The optima from the simulated strategies, denoted by colored dots, are generally concentrated in the vicinity of the true optimal strategies, which indicates that estimation uncertainty does not affect much the portfolio allocation problem. As the sample size $N$ increases, the estimation becomes even more accurate and more mass gathers around the true optima.

The second figure depicts the case of conditioning values at the median and in the upper conditional quantiles of $X_t$. The long strategy, characterized by a share close to 1 invested over very short intervals, is optimal as we move from the center of the distribution towards the bubble zone. Multiple optimal strategies arise as we move towards the inflation phase of the bubble. This comes in hand with different investors betting on different scenarios according to their risk adversity. Investing all wealth in the bubble asset over a 2-period horizon appears to be as

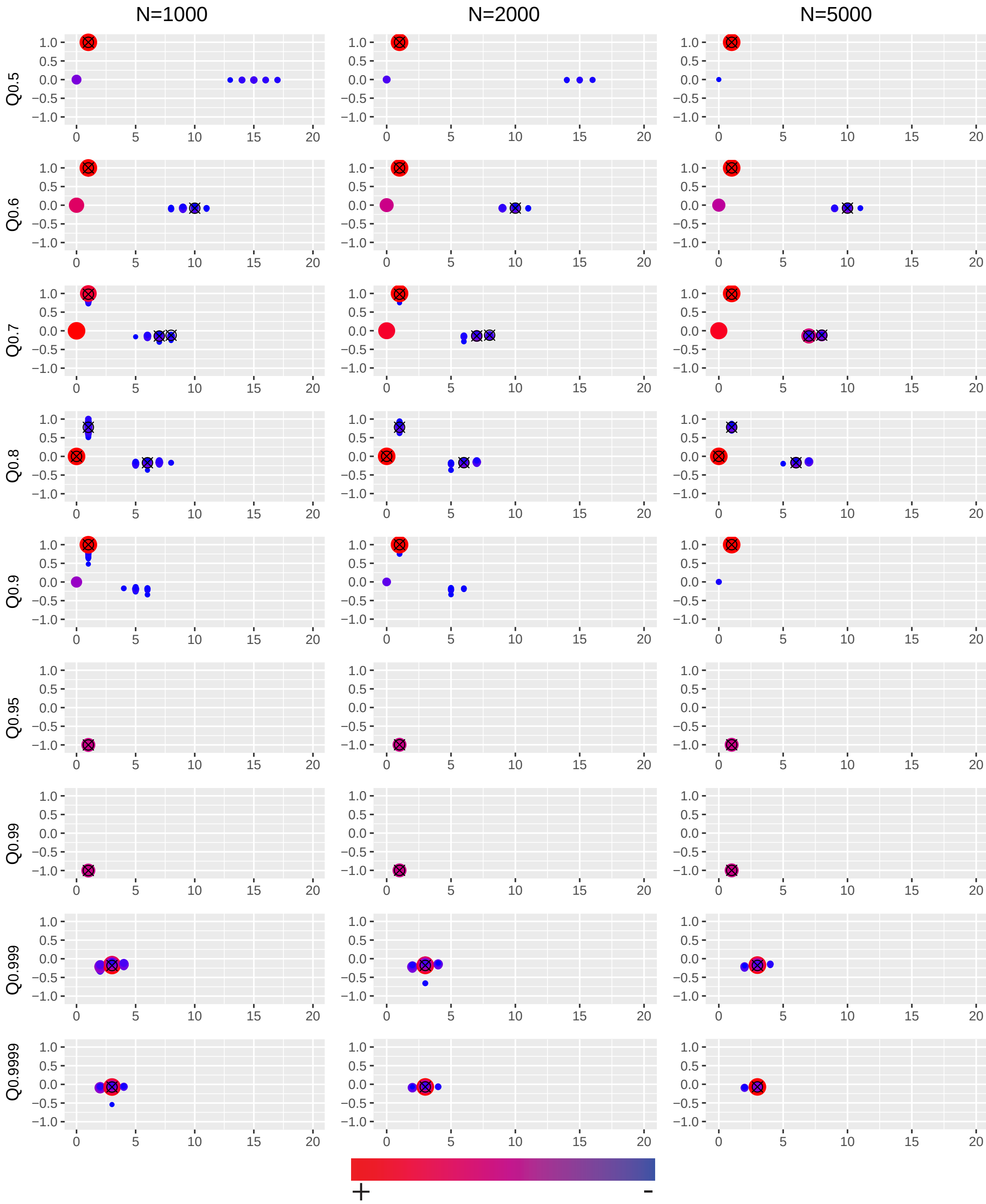
[12] We do not report the precise values as they vary from one subplot to another due to the multiple equilibrium issue discussed earlier and the plot would become too dense to be easily readable.

Figure 2: Optimal portfolio strategies (lower quantiles)




*Notes:* Mass repartition of the optimal portfolio strategies for the CRRA utility function with $\gamma = 10$ when the speculative asset's parameters are estimated by ML across 2000 simulated trajectories of length $N = 1000, 2000, 5000$ trading days and for several starting values defined by the quantiles, $Q_{\cdot}$, of the true marginal distribution of $X_t$. The DGP for the speculative asset price is a MAR(1,1) process $(1-0.9F)(1-0.3B)X_t = \varepsilon_t$ with $\varepsilon_t \overset{i.i.d.}{\sim} S(1.8, 0.5, 0.1, 2)$. The results are displayed in the share (vertical axis) - horizon (horizontal axis) space. The larger and redder the dots, the bigger the proportion of selected portfolios falling in that area across the 2000 simulations. A black target symbol indicates a *true* optimal portfolio, i.e. obtained for the true values of the parameters.

Figure 3: Optimal portfolio strategies (upper quantiles)




*Notes:* Mass repartition of the optimal portfolio strategies for the CRRA utility function with $\gamma = 10$ when the speculative asset's parameters are estimated by ML across 2000 simulated trajectories of length $N = 1000, 2000, 5000$ trading days and for several starting values defined by the quantiles, $Q_{\cdot}$, of the true marginal distribution of $X_t$. The DGP for the speculative asset price is a MAR(1,1) process $(1 - 0.9F)(1 - 0.3B)X_t = \varepsilon_t$ with $\varepsilon_t \overset{i.i.d.}{\sim} S(1.8, 0.5, 0.1, 2)$. The results are displayed in the share (vertical axis) - horizon (horizontal axis) space. The larger and redder the dots, the bigger the proportion of selected portfolios falling in that area across the 2000 simulations. A black target symbol indicates a *true* optimal portfolio, i.e. obtained for the true values of the parameters.

optimal as investing a very small share of wealth in this asset but over a longer horizon, between 7 and 10 periods. Besides, the suboptimal no-investment strategy defined by convention as (0,0) is rarely visited. Next, as the starting value increases and crosses the 0.95 quantile, we get into an explosive regime, where the optimal strategy is to completely short the bubble asset over a one-period horizon. Finally, above the 0.99 quantile, i.e. as the explosive regime becomes more evident, the optimal strategy consists in a fair short position over 3 periods, which is consistent with the increasing bubble crash risk.

Additionally, the dispersion of simulation-based strategies around the true ones rapidly shrinks with the sample size, suggesting that the true optima can indeed be consistently retrieved after parameter estimation. For the quantiles furthest in the tail, the dispersion is in the horizon dimension rather than in the share dimension. Estimation uncertainty on the verge of a bubble crash thus mainly impacts the holding horizon. The results are robust to the choice of the risk-aversion parameter and to changes in the speculative asset price data generating process.[13]

## 4. Economic Value

In this section, we illustrate the usefulness of our approach to provide high-performing portfolio allocation strategies. As discussed previously, in our framework, the optimal investment strategies vary according to the conditioning values $X_t = x$ of the marginal distribution of the process. For this reason, they are expected to outperform standard mean-variance and equally-weighted portfolios, that cannot take into account the current state of the nature at the moment of investing. We study this intuition in the same Monte Carlo setup as in the previous section. More precisely, we generate 1000 trajectories of $N = 2000$ observations from the MAR(1,1) process.

For each trajectory, we use the first two thirds of the data, $\{1, 2, \ldots, T\}$, labeled as in-sample, to estimate the conditional moments of returns and identify the optimal investment strategies in the form of couples $(\omega, h)$ for conditioning values covering the whole marginal distribution of $X_t$ implied by the DGP. The remaining one third of the data, $\{T+1, \ldots, T+k, \ldots N\}$, labeled as out-of-sample, is used as conditioning values for a new investment. This means we assume the investor wishes to invest his wealth in the two assets at a certain date, say $T+k$, within the

[13] The results are qualitatively similar to those obtained when using a non-causal $AR(1)$ as a DGP, but the latter seems to be quite restrictive in practice as it imposes a sudden crash of the bubble. We prefer the more general $MAR(1,1)$ specification and accept a loss in efficiency in the case where the causal parameter should actually be null.

out-of-sample period. To select the optimal share of the bubble asset in the portfolio, $\omega$, and the duration of this risky investment, $h$, out of the $H = 250$ periods of the overall investment, she searches for the closest quantile of the theoretical distribution of $X_t$ just below the actual conditioning price at the selected date. The couple(s) $(\omega, h)$ estimated in-sample for this quantile by using the $CRRA$ utility function with $\gamma = 10$ will then be used to construct the portfolio strateg(y/ies). For each strategy, the portfolio is rebalanced one, at period $T + k + h$. Consisting only in an investment in the no-bubble asset, it is then hold constant up until date $T + k + H$.

For comparison reasons, we compute also the mean-variance and the equally-weighted portfolios over the same periods. In the case of the $MV$ benchmark portfolio, we use the in-sample data to estimate the optimal investment share in the bubble asset. Then, we use it to construct a buy and hold strategy over $H$ periods for each out-of-sample starting date $T + k$. Finally, the computation of the $EW$ portfolio for the same investment horizon is immediate.

To compare the economic value of these strategies we rely on the methods introduced in Subsection 2.3. As our approach may lead to multiple optimal strategies for a given conditioning value, we report three statistics: the average one, labeled $MAR_{mean}$, the one leading to the highest terminal wealth, labeled $MAR_{max}$, and the one leading to the lowest terminal wealth, labeled $MAR_{min}$.

Table 1 reports the results for the five portfolio strategies in terms of average, $\mu$, and standard deviation, $\sigma$, of each performance measure over the 1000 simulated out-of-sample trajectories. Asteriks $(*/*)$ associated with the estimated $\mu$ of each of our strategies indicate that the mean of the performance measure is statistically different from that of the $(EW/MV)$ portfolios according to Wilcoxon's test.

The average wealth for the three $MAR$-based portfolios is similar and always well above that of the benchmark portfolios. Wilcoxon's test always rejects the null of equal averages, suggesting that our approach performs best in terms of terminal wealth. The results are similar when relying on the Sharpe ratio instead of wealth. Notice that the standard deviation is inflated in this case, but it largely diminishes when using the more appropriate modified Sharpe ratio that accounts for higher order moments of portfolio return distribution. Regardless of the measure used, our approach performs significantly better than the EW and MV ones, and this holds even in the worst case scenario, i.e. $MAR_{min}$. The positive averages of the opportunity cost also support these findings. A smaller amount needs to be added to the $MV$ strategy than to the $EW$ one to provide the same expected utility as our $MAR$ strategies.

Table 1: Relative performance of portfolio strategies

| | | Strategy | | | | |
|---|---|---|---|---|---|---|
| | | $MAR_{mean}$ | $MAR_{min}$ | $MAR_{max}$ | $EW$ | $MV$ |
| Wealth | $\mu$ | $1.022^{*/*}$ | $1.019^{*/*}$ | $1.023^{*/*}$ | 1.010 | 1.016 |
| | $\sigma$ | 0.023 | 0.024 | 0.024 | 0.020 | 0.018 |
| Sharpe | $\mu$ | $0.063^{*/*}$ | $0.059^{*/*}$ | $0.065^{*/*}$ | 0.015 | 0.040 |
| | $\sigma$ | 0.064 | 0.065 | 0.064 | 0.020 | 0.030 |
| mSharpe | $\mu$ | $0.085^{*/*}$ | $0.081^{*/*}$ | $0.095^{*/*}$ | 0.035 | 0.064 |
| | $\sigma$ | 0.029 | 0.029 | 0.029 | 0.017 | 0.071 |

| | | $EW$ vs | | | $MV$ vs | | |
|---|---|---|---|---|---|---|---|
| | | $MAR_{mean}$ | $MAR_{min}$ | $MAR_{max}$ | $MAR_{mean}$ | $MAR_{min}$ | $MAR_{max}$ |
| OC | $\mu$ | 0.012 | 0.009 | 0.013 | 0.005 | 0.004 | 0.007 |
| | $\sigma$ | 0.021 | 0.023 | 0.023 | 0.014 | 0.016 | 0.015 |

*Notes:* Our $MAR(1,1)$-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth, Sharpe ratio and modified Sharpe ratio. The opportunity cost (OC) relatively to the two benchmark portfolios is also provided. The results take the form of out-of-sample average and standard deviation over the 1000 simulations. Asteriks $(*/*)$ indicate the rejection of the null hypothesis of Wilcoxon's test of equality of medians at the 95% level relatively to each of the two benchmark strategies, EW and MV, respectively.

As our approach is specifically designed for investors that wish to take advantage of bubble periods, in Table 2 we focus on this setup. We assume that one invests only when the unconditional price process seems to exhibit a locally explosive behaviour, i.e. the conditioning values $X_{T+k} = x$ are above the 95% quantile of the theoretical distribution of the process. The average terminal wealth for our strategies are bigger than in the case when all the marginal distribution is considered, whereas that of the benchmark strategies is lower. The modified Sharpe ratio behaves similarly. The opportunity cost remains positive, relatively constant for the $EW$ strategy and lower than in Table 1 for the $MV$ portfolio. All in all, these results indicate that our method may prove useful for the investor that includes a bubble asset in her portfolio.

The results are robust to eliminating the (0, 0) strategies that were defined by convention, and which do not include the bubble asset at all. They hold when investigating the case of negative bubbles, i.e. looking only at conditioning values beyond the 95% quantile of the theoretical distribution of the process. Finally, they are qualitatively similar when fixing the risk-aversion

Table 2: Relative performance of portfolio strategies in positive bubble period

| | | Strategy | | | | |
|---|---|---|---|---|---|---|
| | | $MAR_{mean}$ | $MAR_{min}$ | $MAR_{max}$ | $EW$ | $MV$ |
| Wealth | $\mu$ | $1.032^{*/*}$ | $1.028^{*/*}$ | $1.034^{*/*}$ | 0.960 | 0.990 |
| | $\sigma$ | 0.047 | 0.046 | 0.048 | 0.030 | 0.020 |
| Sharpe | $\mu$ | $0.053^{*/*}$ | $0.048^{*/*}$ | $0.050^{*/*}$ | -0.024 | 0.002 |
| | $\sigma$ | 0.068 | 0.070 | 0.067 | 0.020 | 0.040 |
| mSharpe | $\mu$ | $0.092^{*/*}$ | $0.085^{*/*}$ | $0.093^{*/*}$ | -0.190 | -0.026 |
| | $\sigma$ | 0.018 | 0.017 | 0.018 | 0.026 | 0.089 |

| | | vs $EW$ | | | vs $MV$ | | |
|---|---|---|---|---|---|---|---|
| | | $MAR_{mean}$ | $MAR_{min}$ | $MAR_{max}$ | $MAR_{mean}$ | $MAR_{min}$ | $MAR_{max}$ |
| OC | $\mu$ | 0.010 | 0.010 | 0.010 | 0.002 | 0.002 | 0.002 |
| | $\sigma$ | 0.021 | 0.021 | 0.021 | 0.019 | 0.019 | 0.019 |

*Notes:* see note to Table 1. The results are based only on the cases where the investment is performed while the first asset exhibits a bubble period, i.e. $X_{T+k} = x$ is beyond the 95% quantile of the theoretical distribution of the price process.

parameter $\gamma$ to 5.

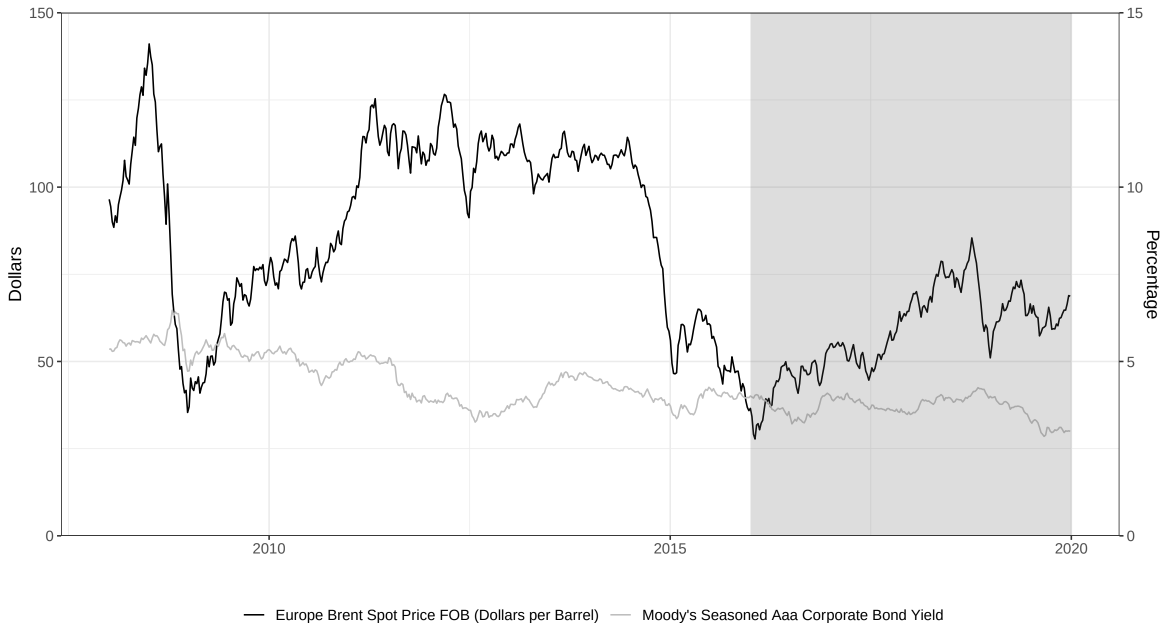
## 5. Empirical illustration

This section illustrates the performance of the proposed portfolio allocation strategy by using Brent oil data for the bubble asset. Weekly data ranging from 2006-01 to 2022-01 have been obtained from EIA and splitted into an in-sample part (2006-01 to 2016-01) and an out-of-sample one (2016-02 to 2020-01). At the same time, we use *Moody's Seasoned Aaa Corporate Bond Yield, Percent, Daily, Not Seasonally Adjusted* from FRED database as the safer asset.

Figure 4 displays the dynamics of the two series, the gray region corresponding to the out-of-sample. The presence of locally explosive behaviour in the Brent data is easy to notice and confirmed by the generalized-sup ADF test of Phillips et al. [2015], see Table 3. At the same time, the Moody's Aaa Corporate Bond yield does not exhibit bubble behaviour according to the same test.

As the oil series seems to exhibit nonstationarity, a trending time varying fundamental part must be extracted before estimating the MAR model on the in-sample data. Two detrending

Figure 4: Data




*Notes: The shaded area corresponds to the out-of-sample data.*

methods have gained interest in this literature. A polynomial function has been used by Hencic and Gouriéroux [2015] and by Hecq and Voisin [2019], while Hecq and Voisin [2021] use the Hodrick- Prescott filter. As the first approach is more direct to implement in the out-of-sample, we follow Hecq and Voisin [2019] and use a polynomial trend of order four to capture trending patterns before estimating the $MAR$ parameters on the detrended series. Besides, we rely on the procedure of Lanne and Saikkonen [2011] based on the AIC information criterion to perform model selection on causal-non-causal models and identify the $MAR(1,1)$ as the best specification. The one-year investment horizon is defined by fixing $H = 52$.

Table 3: GS-ADF Test

| | | Finite Sample Critical Value | | |
|---|---|---|---|---|
| | Test Stat. | 90% | 95% | 99% |
| Brent | 2.62 | 1.95 | 2.17 | 2.58 |
| Moody's Aaa | 0.64 | | | |

*Notes:* Generalized-sup ADF test for the presence of multiple bubbles developed by Phillips et al. [2015]. The critical values are based on 2000 simulations.

Table 4 reports the estimation results. The coefficients of the polynomial trend function are significant, supporting the use of this strategy. Most importantly, the non-causal component dominates the causal one, revealing the forward-looking steady increase in the oil-price data followed by quite abrupt bubble bursts. Next, we compute the quantiles and conditional moments of the detrended series, the latter being fed to the CRRA portfolio optimization program ($\gamma = 10$) to obtain portfolio strateg(y/ies) in the form of couples ($\omega$, $h$).

Table 4: Estimated MAR(1,1) +trend on Brent data

| Detrending method: polynomial of order 4 | | | | | |
|---|---|---|---|---|---|
| Intercept | $\tau^1$ | $\tau^2$ | $\tau^3$ | $\tau^4$ | |
| 1.258e+02$^{*}$ | -1.764e+00$^{**}$ | 1.660e-02$^{**}$ | -5.025e-05$^{***}$ | 4.652e-08$^{***}$ | |
| $\alpha$-stable MAR(1,1) | | | | | |
| $\varphi^{\bullet}$ | $\varphi^{\circ}$ | $\alpha$ | $\beta$ | $\sigma$ | $\mu$ |
| 0.981$^{**}$ | 0.281$^{**}$ | 1.82$^{**}$ | 0.075$^{**}$ | 1.96$^{**}$ | 1.06$^{**}$ |

*Notes:* Estimated parameters of the polynomial trend function and of the $\alpha$-stable MAR(1,1) process associated with the detrended Brent series for the period 2006-2016. Asterisks $^{*}$, $^{**}$, and $^{***}$ indicate significance at the 90%, 95% and 99% level, respectively.

To evaluate the performance of these strategies, we turn to the out-of-sample data. First, we rely on the polynomial coefficient estimates to extend the trend dynamics and remove it from the data. This allows us to match each out-of-sample detrended value that plays the role of a conditioning price, $X^d_{T+k}$, with the closest floor empirical in-sample quantile of the detrended series and identify the associated portfolio allocation strateg(y/ies). The associated out-of-sample terminal wealth is computed by applying the couple(s) ($\omega$, $h$) to the out-of-sample non-detrended price data.

The results are reported in Table 5. As in Tables 1-2, they display the average and the standard deviation of various performance measures over the out-of-sample period for each of the five portfolio strategies. Asteriks $(*/*)$ associated with the estimated $\mu$ of each of our strategies indicate that the median terminal wealth is statistically different from that of the $(EW/MV)$ portfolios according to Wilcoxon's test. To compare the Sharpe and the mSharpe ratios of our approach to those of the benchmark portfolios, we rely on the tests of Ardia and Boudt [2015]

Table 5: Relative performance of portfolio strategies for Brent

| | | Strategy | | | | |
|---|---|---|---|---|---|---|
| | | $MAR_{mean}$ | $MAR_{min}$ | $MAR_{max}$ | $EW$ | $MV$ |
| Wealth | $\mu$ | $1.007^{*/*}$ | 0.998 | $1.019^{*/*}$ | 0.995 | 0.993 |
| | $\sigma$ | 0.067 | 0.077 | 0.094 | 0.065 | 0.064 |
| Sharpe | $\mu$ | $0.058^{*/*}$ | -0.032 | $0.065^{*/*}$ | -0.019 | -0.050 |
| | $\sigma$ | 0.45 | 0.45 | 0.42 | 0.43 | 0.41 |
| mSharpe | $\mu$ | $0.218^{*/*}$ | $0.178^{/*}$ | $0.267^{*/*}$ | 0.195 | 0.150 |
| | $\sigma$ | 0.55 | 0.78 | 0.80 | 0.81 | 0.86 |

| | | $EW$ vs | | | $MV$ vs | | |
|---|---|---|---|---|---|---|---|
| | | $MAR_{mean}$ | $MAR_{min}$ | $MAR_{max}$ | $MAR_{mean}$ | $MAR_{min}$ | $MAR_{max}$ |
| OC | $\mu$ | 0.012 | 0.012 | 0.019 | 0.0021 | -0.005 | 0.026 |
| | $\sigma$ | 0.043 | 0.062 | 0.065 | 0.021 | 0.041 | 0.081 |

*Notes:* Our $MAR(1,1)$-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth, Sharpe ratio and modified Sharpe ratio. The opportunity cost (OC) relatively to the two benchmark portfolios is also provided. The results take the form of out-of-sample average and standard deviation. Wilkoxon's test is used for the terminal wealth and we rely on the tests by Ardia and Boudt [2015] for the (m)Sharpe ratios. Asteriks $(*/*)$ indicate the rejection of the null hypothesis of each test at the 95% level.

Table 6: Relative performance of portfolio strategies for Brent in positive bubble period

| | | Strategy | | | | |
|---|---|---|---|---|---|---|
| | | $MAR_{mean}$ | $MAR_{min}$ | $MAR_{max}$ | $EW$ | $MV$ |
| Wealth | $\mu$ | $0.996^{*/*}$ | $0.996^{*/*}$ | $0.996^{*/*}$ | 0.966 | 0.965 |
| | $\sigma$ | 0.064 | 0.064 | 0.064 | 0.037 | 0.035 |
| Sharpe | $\mu$ | $-0.005^{*/*}$ | -0.010 | $0.035^{*/*}$ | -0.088 | -0.100 |
| | $\sigma$ | 0.50 | 0.47 | 0.50 | 0.41 | 0.46 |
| mSharpe | $\mu$ | $0.188^{*/*}$ | $0.185^{*/*}$ | $0.297^{*/*}$ | 0.148 | 0.169 |
| | $\sigma$ | 0.95 | 0.90 | 0.93 | 0.50 | 1.06 |

| | | $EW$ vs | | | $MV$ vs | | |
|---|---|---|---|---|---|---|---|
| | | $MAR_{mean}$ | $MAR_{min}$ | $MAR_{max}$ | $MAR_{mean}$ | $MAR_{min}$ | $MAR_{max}$ |
| OC | $\mu$ | 0.021 | 0.014 | 0.032 | 0.001 | -0.01 | 0.038 |
| | $\sigma$ | 0.05 | 0.08 | 0.08 | 0.02 | 0.05 | 0.10 |

*Notes:* see note to Table 5. The results are based only on the cases where the investment is performed while the Brent exhibits a bubble period, i.e. $X_{T+k} = x$ is beyond the 95% quantile estimated with the in-sample data.

with asymptotic HAC standard errors.[14].

The results indicate a positive gain in using our approach relatively to the standard $MV$ and $EW$ portfolios. Indeed, the terminal wealth indicates a return on investment of about 2% for our $MAR_{max}$ and 0.7% for the $MAR_{mean}$, whereas the benchmark strategies are loosing more than our $MAR_{min}$. The (m)Sharpe ratios are always higher for the $MAR$ strategies, while the opportunity cost is positive. Besides, the comparison tests generally indicate a significant difference in the average performance which is in favor of our approach. All results hold when focusing only on the case of a positive bubble period, i.e. conditioning values beyond the 95% quantile. These are displayed in Table 6.

## 6. Conclusion

In this paper we propose an asset allocation strategy particularly designed for the case of an investors wishing to include a bubble asset in his/her portfolio. For this, we account explicitly for the distributional characteristics of bubble assets through a MAR(1,1) model, which seems to be appropriate to capture locally explosive behaviours. The higher-order conditional moments of the return distribution are then plugged in the Taylor-series-expansion of the CRRA utility function and the PGP algorithm, respectively. The economic value of our strategy is compared in out-of-sample with standard benchmarks such as the mean-variance and equally-weighted portfolios based on well-known performance measures such as the opportunity cost, the Sharpe ratio and the modified Sharpe ratio. Both simulation-based analyses and empirical results using Brent data support the superiority of our approach. This is particularly the case when the conditioning values $X_t = x$ are in the tail of the marginal distribution of the process, meaning that the bubble is growing and approaching its peak. This is of outmost importance for investors, as the risk of bubble crash is the highest.

[14]The results are similar when the i.i.d. bootstrap approach is used to compute the standard errors.

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## Appendix A. Proofs

*Appendix A.1. Conditional moments of returns*

For the bubble asset we have the non-central moments of returns

$$\mathbb{E}(r^{X^j}_{t,t+h}|X_t) = \frac{1}{X_t}\mathbb{E}(X^p_{t,t+h}|X_t), \quad \text{for} \quad p \geq 1,$$

where the conditional moments of the price series are defined through Proposition 1.

For the second asset, using the properties of the geometric brownian motion, one obtains

$$\begin{aligned}
\mathbb{E}(r^{S}_{t,t+h}|X_t) &= e^{\upsilon h} - 1 \\
\mathbb{E}(r^{S^2}_{t,t+h}|X_t) &= e^{2\upsilon h + \varsigma^2 h} - 2e^{\upsilon h} + 1 \\
\mathbb{E}(r^{S^3}_{t,t+h}|X_t) &= e^{3\upsilon h + 3\varsigma^2 h} - 3e^{2\upsilon h + \varsigma^2 h} + 3e^{\upsilon h} - 1 \\
\mathbb{E}(r^{S^4}_{t,t+h}|X_t) &= e^{4\upsilon h + 4\varsigma^2 h} - 4e^{3\upsilon h + 3\varsigma^2 h} + 6e^{2\upsilon h + \varsigma^2 h} - 4e^{\upsilon h} + 1,
\end{aligned}$$

and similar expressions, using $H-h$ instead of $h$, can be derived for $\mathbb{E}(r^S_{t+h,t+H}|X_t)$. Making use of the mapping relations between central and non-central moments and the independence between the two assets, one can subsequently express $\mathbb{E}\left[(R_{t+H}-\overline{R}_{t+H})^k|X_t,S_t\right]$ as a function of these moments.

*Appendix A.2. Derivation of the constants $\sigma_1^\alpha$, $\beta_1$, $\kappa_p$, and $\lambda_p$*

Fries 2021 shows that if $X_t$ is a $\alpha$-stable two-sided MA($\infty$) process with $0<\alpha<2, \alpha\neq 1$, $\beta\in[-1,1]$, and $\sigma>0$ as defined in Section 2.2, i.e. well defined, stationary process with $\alpha$-stable errors, and for $h\geq 1$ then one can obtain the conditional moments of the process $X_t$ for $p\leq 4$ with

$$\sigma_1=\sigma^\alpha\sum_{k\in\mathbb{Z}}|a_k|^\alpha, \qquad \beta_1=\beta\frac{\sum_{k\in\mathbb{Z}}a_k^{<\alpha>}}{\sum_{k\in\mathbb{Z}}|a_k|^\alpha}, \qquad \kappa_p=\frac{\sum_{k\in\mathbb{Z}}|a_k|^\alpha(\frac{a_{k-h}}{a_k})^p}{\sum_{k\in\mathbb{Z}}|a_k|^\alpha}, \qquad \lambda_p=\frac{\sum_{k\in\mathbb{Z}}a_k^{<\alpha>}(\frac{a_{k-h}}{a_k})^p}{\sum_{k\in\mathbb{Z}}|a_k|^\alpha},$$

where $y^{<\alpha>}=\text{sign}(y)|y|^\alpha$ for any $y\in\mathbb{R}$. Using his results together with the fact that the coefficients of the $MA(\infty)$ representation of a MAR(1,1) process, $X_t=\sum_{k=-\infty}^{\infty}a_k\varepsilon_{t-k}$, satisfy

$$\begin{aligned}
a_k &= \frac{\varphi^{\circ k}}{1-\varphi^\circ\varphi^\bullet} \qquad \text{if } k\geq 0,\\
a_k &= \frac{\varphi^{\bullet -k}}{1-\varphi^\circ\varphi^\bullet} \qquad \text{otherwise,}
\end{aligned}$$

and calculus based on geometric series, one can easily obtain the results in Proposition 1.

## Appendix B. PGP optimisation framework

We follow Aksarayah and Pala (2018) to implement the PGP model based on the first four conditional moments of the return distribution. This approach is not subjected to a Taylor approximation (as the CRRA utility function case), but the weights it attributes to the portfolio moments cannot be precisely related to the parameters of a utility function. In this case, the wealth $W_t$ of the investor is allocated at time $t$ by solving conflicted multi objectives such as maximizing expected return and skewness and minimizing variance and kurtosis that are weighted with investor preferences. The PGP model can be defined as

$$\min_{(\omega,h)} \Big(1+|d_1-R^*|\Big)^{\gamma_1}+\Big(1+|d_2-V^*|\Big)^{\gamma_2}+\Big(1+|d_3-S^*|\Big)^{\gamma_3}+\Big(1+|d_4-K^*|\Big)^{\gamma_4}, \tag{B.1}$$

$$\text{s.t. } R_{(\omega,h)}+d1=R^*,\;\; V_{(\omega,h)}-d2=V^*,\;\; S_{(\omega,h)}+d3=S^*,\;\; K_{(\omega,h)}-d4=K^*,\;\; d_i\geq 0,$$

where $R_{(\omega,h)}$, $V_{(\omega,h)}$, $S_{(\omega,h)}$, and $K_{(\omega,h)}$ denote respectively the expectation, variance, skewness and excess kurtosis of the returns $R_{t+H}$ conditional on the price level $X_t = x$ for a given strategy $(\omega, h)$ defined as

$$R_{\omega,h} := \mathbb{E}\Big[R_{t+H}|X_t,S_t\Big],\;\; V_{\omega,h} := \mathbb{E}\Big[(R_{t+H}-R_{\omega,h})^2|X_t,S_t\Big],$$
$$S_{\omega,h} := \mathbb{E}\Big[(R_{t+H}-R_{\omega,h})^3/V_{\omega,h}^{3/2}|X_t,S_t\Big],\;\; K_{\omega,h} := \mathbb{E}\Big[(R_{t+H}-R_{\omega,h})^4/V_{\omega,h}^{2}|X_t,S_t\Big]-3,$$

Besides, $R^*, V^*, S^*, K^*$ denote the optima of the subprograms $\max_{(\omega,h)} R_{(\omega,h)}$, $\min_{(\omega,h)} V_{(\omega,h)}$, $\max_{(\omega,h)} S_{(\omega,h)}$, $\min_{(\omega,h)} K_{(\omega,h)}$, and the $\gamma_i$'s are non-negative parameters weighting the preference of the investor to pursue optimality of one moment over the others.