FORECAST PERFORMANCE OF NON-CAUSAL AUTOREGRESSIONS AND THE IMPORTANCE OF UNIT ROOT PRETESTING *

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Abstract

Based on large simulations study, this paper investigates which strategy to adopt in order to choose the best forecasting model — in terms of accuracy — for Mixed causal-noncausal AutoRegressions (MAR) data generating processes: always differencing (D), never differencing (L) or unit root pretesting (P). Relying on recent econometric developments regarding forecasting and unit root testing in this MAR framework, the main results suggest that from a practitioner's point of view, the P strategy at the 1%-level is a good compromise. In fact, it never departs too much from the best model in terms of forecast accuracy, unlike the L (respectively D) strategy when the DGP becomes very persistent (respectively with little persistence).

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1 Introduction

Regarding time series forecasting, the question about using a model for the series in levels or in first differences can be traced back to Box and Jenkins [1976]. In their popular book, these authors recommend to use the model in levels unless the root of the process to forecast is close to unity, in which case the model in first differences would achieve the best forecast accuracy. The introduction of unit root tests by Dickey and Fuller [1979] and their application in an influential paper by Nelson and Plosser [1982] gave rise to a renewed interest in this question. Indeed, from a large set of long historical time series for the U.S. economy, Nelson and Plosser [1982] conclude that the null hypothesis that these series contain a unit root cannot be rejected for most of them. Yet, as emphasized in Dickey, Bell, and Miller [1986], forecasts of a unit root process are very different from forecasts of a stationary process, also when the latter is strongly persistent. Hence, beside "always difference" or "never difference" (denoted respectively D and L hereafter) forecasting strategies, a third option has emerged: unit-root pretesting (P) and difference or not accordingly. From Monte-Carlo simulations, evidence in favour of pretesting strategy was found for the linear autoregressive class of models by e.g. Campbell and Perron [1991], Stock [1996], Stock and Watson [1999] or Diebold and Kilian [2000], especially for roots close to unity.

In this paper, we answer the same question for the class of mixed causal-noncausal autoregressive (MAR) models. These models, introduced decades ago in statistics, have acquired growing popularity in economics and finance over the past few years. Development of their econometric theory as well as applications to macroeconomic or financial time series modelling and forecasting can be found in Lanne and Saikkonen [2011], Lanne, Luoma, and Luoto [2012], Lanne and Saikkonen [2013], Hencic and Gouriéroux [2015], Gouriéroux and Zakoian [2015], Gouriéroux and Jasiak [2016], Gouriéroux and Zakoian [2017], Fries and Zakoian [2019], Hecq and Voisin [2020] or Cavaliere, Nielsen, and Rahbek [2020], among others. The interest in MAR models, which allow for dependence on both the past and the future — unlike the well-known backward-looking autoregression which rules out dependence on future observations — stems mainly from three of their characteristics. First, they are able to capture epochs of bubble build-up and burst. Second, they can account for non-fundamental shocks which can reflect, for instance, the fact that the econometrician uses a smaller information set than economic agents do. Third, as noticed in a previous paper [Bec, Nielsen, and Saïdi, 2020], mixed causal-noncausal autoregressive models could prove very useful for forecasting as they can be viewed as very parsimonious representations of more complex nonlinear DGPs (see also Gouriéroux and Zakoian [2017]

on this point). Due to their dependence on future observations, specific forecasting methods have been developed for MAR models by Lanne, Luoto, and Saikkonen [2012] and Gouriéroux and Jasiak [2016]. Similarly, a specific unit root test has been proposed by Saikkonen and Sandberg [2016].

The goal of this paper is to evaluate the relevance of unit root pretesting (P) for these MAR models' forecasting performance, compared to the D and L strategies. To this end, the recent developments cited above are used in a large simulation study. Within this methodological framework, a variety of degrees of persistence, of forecast horizons, of sample sizes as well as of levels of the test are under scrutiny.

After a brief presentation of the MAR model in Section 2, Section 3 exposes the amended version of Saikkonen and Sandberg [2016] unit root test proposed in a previous work [Bec, Nielsen, and Saïdi, 2020] in order to circumvent likelihood bimodality issues involved by the estimation of the MAR models. Then, Section 4 describes the forecasting methods of Lanne, Luoto, and Saikkonen [2012] and Gouriéroux and Jasiak [2016]. Section 5 reports results of preliminary simulation exercises which are used to fine-tune the settings of the main simulation study reported in Section 6. Section 7 concludes.

2 The MAR Setting

Consider the mixed causal non-causal autoregression, MAR(1,1), as given by

$$\Phi(L)\Psi(L^{-1})y_t = (1-\phi L)(1-\psi L^{-1})y_t = \epsilon_t, \tag{1}$$

where ϵ_t is assumed i.i.d. with p.d.f. given by $g(\cdot;\theta)$ indexed by parameters in θ. We consider the case $-1 < \phi \leq 1$ and $-1 < \psi < 1$, allowing a unit root in the causal polynomial, $\Phi(L)$, see Saikkonen and Sandberg [2016] for a discussion.

Following Lanne and Saikkonen [2011] and Gouriéroux and Jasiak [2016], define the unobserved causal and non-causal components as, respectively,

$$u_t = (1-\phi L)y_t \quad \text{and} \quad v_t = (1-\psi L^{-1})y_t, \tag{2}$$

such that $(1-\psi L^{-1})u_t = \epsilon_t$ and

$$u_t = (1-\psi L^{-1})^{-1}\epsilon_t = \sum_{j=0}^{\infty} \psi^j \epsilon_{t+j}. \tag{3}$$

For a specified innovation density $g(\cdot;\theta)$, the parameters in (1) can be estimated from a sample $\{y_t\}_{t=1}^T$ using approximate maximum likelihood,

$$(\hat{\phi},\hat{\psi},\hat{\theta}) = \arg\max_{\phi,\psi,\theta} \sum_{t=2}^{T-1} \log g\left((1-\phi L)(1-\psi L^{-1})y_t;\theta\right),$$

conditional on initial and terminal values, y_1 and y_T, see e.g. Breidt et al. [1991], and Lanne and Saikkonen [2011]. Below, we use a Student's $t(0,\sigma^2,\lambda)$ distribution, such that

$$g(\epsilon_t;\sigma,\lambda) = \frac{\Gamma(\frac{\lambda+1}{2})}{\Gamma(\frac{\lambda}{2})}\left(\pi\lambda\sigma^2\right)^{-\frac{1}{2}}\left(1+\frac{\epsilon_t^2}{\sigma^2\lambda}\right)^{-\frac{\lambda+1}{2}}, \tag{4}$$

with $\theta = (\lambda,\sigma^2)$ containing the degrees of freedom parameter, λ, and the scale, σ^2.

3 Unit Root Testing

The unit root test pertaining to the hypothesis $\mathcal{H}_0 : \phi = 1$ in the MAR model is considered in Saikkonen and Sandberg [2016]. They show that for $g(\cdot;\theta)$ symmetric, the usual one-sided unit-root test statistic,

$$\tau = \frac{(\hat{\phi}-1)}{se(\hat{\phi})}, \tag{5}$$

where $se(\hat{\phi})$ is the square root of the relevant entry in the inverse observed information, has a limiting distribution, $D_\tau(\rho)$, that depends on the nuisance parameter ρ. If $g(\cdot;\theta)$ is the Student's density in (4), with $\lambda > 2$, it holds that ρ is a function of λ:

$$\rho(\lambda)^2 = \frac{\lambda(\lambda+1)}{(\lambda-2)(\lambda+3)}.$$

However, Bec et al. [2020] show that a bimodality issue occurs in the Student's t case and leads to interchanged roots. Indeed, the backward root can be estimated as the forward root and *vice versa*. Since the unit root test is based on the estimation of the backward root, this can pose a problem. As a consequence, the estimation strategy proposed by Bec et al. [2020] relies on a grid search procedure in order to characterize the entire likelihood surface and list all local maxima. Then, if there are multiple maxima, the maximum with the backward root higher than the forward root ($\hat{\phi} > \hat{\varphi}$) is selected even

if there exists a maximum with a higher likelihood but with $\hat{\phi} < \hat{\varphi} \approx 1$. This choice stems from the fact that a unit root in the forward component would lack reasonable economic interpretation: It would mean that agents look into the infinite future without discounting. Beside, the test statistic is similar to Saikkonen and Sandberg [2016]'s. Critical values are given in Saikkonen and Sandberg [2016] in terms of a response surface approximation.

4 Forecasting

To forecast $\{y_{T+h}\}_{h=1}^{H}$ based on the MAR model, we consider the approaches suggested in Lanne, Luoto, and Saikkonen [2012] and Gouriéroux and Jasiak [2016]. Lanne et al. [2012] start from (3) and simulate future paths $\{\epsilon_{T+k}\}_{k=1}^{K}$ to approximate the conditional expectation $E(y_{T+h} \mid y_T)$ and the c.d.f. Gouriéroux and Jasiak [2016], on the other hand, derive the predictive density $p(u_{T+1}, ..., u_{T+H} \mid y_T)$ and use importance sampling to draw from the predictive distibution of $\{y_{T+h}\}_{h=1}^{H}$ given y_T.

4.1 Lanne, Luoto, and Saikkonen [2012] (LLS)

To derive the point forecast, it is used that

$$E(y_{T+h} \mid y_T) = \phi^h y_T + E(\sum_{i=0}^{h-1} \phi^i u_{T+i+1} \mid y_T), \quad h = 1, 2, ..., H.$$

The infinite sum in (3) is then approximated with a truncated version using K terms, and based on N simulated paths, $\{\epsilon_{T+k}^{(i)}\}_{k=1}^{K}$ for $i = 1, 2, ..., N$, Lanne et al. [2012] suggests to estimate the conditional expectation as

$$\begin{aligned}
& E(y_{T+h} \mid y_T) \\
= \;& \phi^h y_T + E(\sum\nolimits_{i=0}^{h-1} \phi^i u_{T+i+1} \mid y_T) \\
\approx \;& \phi^h y_T + E\left(\sum\nolimits_{i=0}^{h-1} \phi^i \sum\nolimits_{k=0}^{K-h} \psi^k \epsilon_{t+h+i+k} \mid y_T\right) \\
\approx \;& \phi^h y_T + \frac{\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{i=0}^{h-1}\phi^i \sum_{k=0}^{K-h}\psi^k \epsilon_{t+h+i+k}\right) g\left(\epsilon_T(u_T, \epsilon_{T+1}^{(i)}, ..., \epsilon_{T+M}^{(i)}); \lambda\right)}{\frac{1}{N}\sum_{i=1}^{N} g\left(\epsilon_T(u_T, \epsilon_{T+1}^{(i)}, ..., \epsilon_{T+M}^{(i)}); \lambda\right)}
\end{aligned} \tag{6}$$

where ϵ_T is found as

$$\epsilon_T(u_T, \epsilon_{T+1}^{(i)}, ..., \epsilon_{T+M}^{(i)}) = \hat{u}_T - \sum_{k=1}^{K} \psi^k \epsilon_{T+k}^{(i)}. \tag{7}$$

To get also interval forecasts, we evaluate the c.d.f. over a grid $x \in (x_1, ..., x_G)$ using

$$\frac{\frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{I}\left(\sum_{i=0}^{h-1}\phi^i\sum_{k=0}^{K-h}\psi^k\epsilon_{t+h+i+k}\leq x\right)\right)g\left(\epsilon_T(u_T,\epsilon_{T+1}^{(i)},...,\epsilon_{T+M}^{(i)});\lambda\right)}{\frac{1}{N}\sum_{i=1}^{N}g\left(\epsilon_T(u_T,\epsilon_{T+1}^{(i)},...,\epsilon_{T+M}^{(i)});\lambda\right)},$$

similar to (6) where $\mathbb{I}(\cdot)$ is the indicator function.

In the implementation below we choose $K \in [K_{\min}, K_{\max}]$, such that $\psi^K < 0.0001$ with $K_{\min} = H+5$ and $K_{\max} = 200$. For the interval forecast, we use a grid of 1024 equally spaced points in the interval between $\min(\mu_u - 12\sigma_u, u_T - \sigma_u)$ and $\max(\mu_u + 12\sigma_u, u_T + \sigma_u)$, where μ_u and σ_u^2 denote the empirical mean and variance of $\{u_t\}_{t=1}^T$. These choices ensure that the interval is wide enough to include most probability mass of the forecast distributions for $H = 10$. Specific quantiles are found using linear interpolation.

4.2 Gouriéroux and Jasiak [2016]

To obtain density forecasts for $\{y_{T+h}\}_{h=1}^H$ conditional on y_T, Gouriéroux and Jasiak [2016] suggest to forecast the non-causal component $\{u_{T+h}\}_{h=1}^H$ given y_T, and use (2) to construct density forecasts for $\{y_{T+h}\}_{h=1}^H$ from forecast densities for $\{u_{T+h}\}_{h=1}^H$ and y_T.

To forecast $\{u_{T+h}\}_{h=1}^H$, Gouriéroux and Jasiak [2016] rewrite the joint predictive density,

$$\begin{aligned} p(u_{T+1},...,u_{T+H} \mid y_T) &= p(u_{T+1},...,u_{T+H} \mid u_T) \\ &= \frac{p(u_T,u_{T+1},...,u_{T+H})}{p(u_T)} \\ &= \frac{p(u_T,u_{T+1},...,u_{T+H-1} \mid u_{T+H})p(u_{T+H})}{p(u_T)}. \end{aligned}$$

Using estimated versions for $\{u_t\}_{t=1}^T$, ψ, and θ and estimating the stationary distributions by sample averages, an estimate of the predictive density is given in closed form

$$\begin{aligned} &\hat{p}(u_{T+1},u_{T+2},...,u_{T+H} \mid \hat{u}_T) \\ =\ & g(u_T - \hat{\psi}u_{T+1};\hat{\theta})\cdot g(u_{T+1} - \hat{\psi}u_{T+2};\hat{\theta})\cdots g(u_{T+H-1} - \hat{\psi}u_{T+H};\hat{\theta}) \\ & \times \frac{\sum_{t=1}^T g(u_{T+H} - \hat{\psi}\hat{u}_t;\hat{\theta})}{\sum_{t=1}^T g(u_T - \hat{\psi}\hat{u}_t;\hat{\theta})}. \end{aligned} \tag{8}$$

The closed form for the joint predictive density in (8) can be used to simulate paths for $\{u_{T+h}\}_{h=1}^H$ e.g. using Sampling-Importance-Resampling (SIR), see Rubin [1988].

4.2.1 Implementation

Below, we implement SIR based on an importance density, defined by a causal AR(1) with Student's $t(0, \sigma^2, \lambda^*)$ innovations.[1] We generate candidate forecast paths $\{u_{T+h}^{*(i)}\}_{h=1}^{H}$, for $i = 1, 2, ..., M$, conditional on $\hat{u}_T$, using the causal AR(1) model

$$u_{T+h}^{*(i)} = \gamma^* u_{T+h-1}^{*(i)} + \sigma^* \eta_{T+h}^{*(i)}, \quad \eta_{T+h}^{*(i)} \sim t(0, 1, \lambda^*), \tag{9}$$

with $u_T^{*(i)} = \hat{u}_T$, and use the importance density

$$\hat{q}(u_{T+1}^{*(i)}, ..., u_{T+H}^{*(i)} \mid \hat{u}_T) = g^*(u_{T+H}^{*(i)} \mid u_{T+H-1}^{*(i)}) \cdots g^*(u_{T+2}^{*(i)} \mid u_{T+1}^{*(i)}) g^*(u_{T+1}^{*(i)} \mid \hat{u}_T),$$

where $g^*(\cdot)$ is the conditional density for the causal AR(1) model depending on $(\gamma^*, \sigma^*, \lambda^*)$.

For each candidate path, $\{u_{T+h}^{*(i)}\}_{h=1}^{H}$, we calculate the corresponding importance weight

$$w^{(i)} = \frac{\hat{p}(u_{T+1}^{*(i)}, ..., u_{T+H}^{*(i)} \mid \hat{u}_T)}{\hat{q}(u_{T+1}^{*(i)}, ..., u_{T+H}^{*(i)} \mid \hat{u}_T)}.$$

In the resampling step of the SIR algorithm, we draw N forecast paths from the M candidate paths $\{u_{T+h}^{*(i)}\}_{h=1}^{H}$ $i = 1, 2, ..., M$, with probabilities given by the normalized importance weights

$$p^{(i)} = \frac{w^{(i)}}{\sum_{i=1}^{M} w^{(i)}}.$$

For M and N large, the final sample paths, $\{u_{T+h}^{(j)}\}_{h=1}^{H}$ $j = 1, ..., N$, are draws from the predictive distribution with p.d.f. given by (8).

For the one-step forecast, $h = 1$, the predictive density of $y_{T+1}|y_T$ is based on the empirical distribution of $u_{T+1}^{(j)}$ shifted by ϕy_T, and the distribution of the forecast is represented by $y_{T+1}^{(j)}$ for $j = 1, ..., N$, with $y_{T+1}^{(j)} = u_{T+h}^{(j)} + \hat{\phi} y_T$.

[1] We have also considered a recursive version of the SIR algorithm, where we resample the pairs $\{y_{T+h-1}^{(j)}, u_{T+h}^{(j)}\}$ at each forecast horizon, $h = 1, 2, ..., H$, but otherwise use the same approach as below. Even though this is computationally more demanding, it could have improved the precision because it does not rely on the fit of entire forecast paths. Unfortunately, 10,000 Mote Carlo replications of a MAR(1,1) DGP with $\phi = 0.9$, $\psi = 0.6$, $\sigma = 1$, $\lambda = 7$ and $T = 200$ has revealed no improvement of the forecasts accuracy compared to the SIR approach of Gouriéroux and Jasiak [2016] or the LLS method. These additional results are available upon request.

For longer forecasting horizons, $h = 2, 3, ..., H$, the forecast paths are updated recursively

$$y_{T+h}^{(j)} = u_{T+h}^{(j)} + \phi y_{T+h-1}^{(j)}.$$

Point forecasts, denoted $\{\hat{y}_{T+h}\}_{h=1}^{H}$, can be derived using a location measure for the simulated predictive distribution. Gouriéroux and Jasiak [2016] suggest the pointwise *mode* as the most likely outcome; an alternative would be the *median* or the *mean*.

The effectiveness of the SIR algorithm depends on the chosen importance density, and before discussing the effect of unit root pretesting we consider a small pilot simulation, where we vary $(\gamma^*, \sigma^{2*}, \lambda^*)$ to ensure a sufficiently large variation of the candidate paths.

5 Preliminary Simulations

This Section presents the pilot simulations study which aims to guide the choice of *(i)* the importance density of the SIR algorithm, *(ii)* the location measure for the simulated predictive distribution and *(iii)* the forecasting approach[2].

In the simulation below we use B simulated MAR(1,1) time series, $\{y_t^{(b)}\}_{t=1}^{T+H}$ $b = 1, 2, ..., B$, with sample length $T = 200$, and parameters given by $\phi = 0.9$ and $\psi = 0.6$.

To compare point forecasts, we use the mean absolute deviation from the true value of different implementations for each forecast horizon $h \in \{1, 2, 5, 10\}$, defined as

$$MAD(h) = \frac{1}{B} \sum_{b=1}^{B} \left| \hat{y}_{T+h}^{(b)} - y_{T+h}^{(b)} \right|.$$

We also consider the 90 percent coverage of the density forecasts, i.e. the proportion of cases where the actual value of the time series, y_{t+h}, is included in the 90 percent confidence interval of the forecast. For the pilot simulations, we use $M = N = 10000$ and $B = 10000$ replications unless otherwise mentioned.

[2]We have also explored the influence of the number of iterations. To this end, we have compared forecasts obtained for numbers of simulations, M and N, taking values in $\{50000, 20000, 10000, 5000, 1000\}$. Point forecasts appear to be reliable for quite low number of simulations. For the chosen setting, interval forecasts appear reliable with values of N and M larger than 5000-10000. Accordingly, the subsequent simulation study is conducted for $M = N = 10000$. These additional results are available upon request.

In order to choose the importance density of the SIR algorithm, we conduct a simulation study for various values of $(\gamma^*, \sigma^{2*}, \lambda^*)$. First, we consider $\gamma^* \in \{\tilde{\gamma}, 1\}$, where $\tilde{\gamma}$ is the OLS estimate obtained from a causal AR(1) model for $\{\hat{u}_t\}_{t=1}^T$. For the scale we consider $\sigma^{2*} = c\tilde{\sigma}^2$, with $\tilde{\sigma}^2$ is the OLS estimate and where $c \in \{1, 4\}$ potentially inflates the variance. Finally, we take $\lambda^* \in \{\hat{\lambda}, 3, 100\}$, where $\hat{\lambda}$ is the MLE from the MAR model. The forecast accuracy performances of these various calibrations, as measured by the MAD(h) and the 90%-coverage, are reported in Table 1 in the appendix (columns (1) to (8)), and the following preliminary conclusions emerge:

1. Regarding the location measure, it appears that the mode is always dominated by the mean and the median, the last two ones giving very similar MAD's. This could be due to the well-known noise of the mode estimates. We have considered different versions of mode estimates, and for the present case the maximum of the estimated kernel density (using a Gaussian kernel with 128 steps to 2 percent trimmed data) performs better than the half-sample median, see e.g. Bickel and Frühwirth [2006]. In the subsequent simulation study, we will focus on the *median* because the *mean* does not exist if $\lambda < 2$.

2. The coverage rates, given in the last four lines of Table 1, are generally excellent indicating that the density forecast may not be too sensitive to the precise implementation.

3. The differences between MAD based on different importance densities are generally small for the present case. Overall, the preferred is case (5) based on a unit root in the candidate model given in Equation (9): this is the one retained for the subsequent simulation study.

4. The last column of Table 1 reports the forecast performance based on LLS approach. It is worth noticing that the differences between the MAD's and 90%-coverages obtained from LLS and Gouriéroux and Jasiak [2016] methods are generally very small. Hence, for the purpose followed in this paper, the latter is retained in the subsequent work.

6 Main simulation study

Tables 2 to 5 in the appendix compare the forecast accuracy reached for $T \in \{100, 200, 400\}$ by five different strategies for the choice of the forecasting model: the one called L refers to the model in levels; the next three ones correspond to the pretest P strategy at the 1%-, 5%- and 10%-levels respectively; the last one, D, corresponds to the model in first differences. In this

baseline MAR(1,1) DGP, the forward root parameter is fixed to $\psi = 0.6$ while the degrees of freedom of the Student distribution is set to $\lambda = 7$. Finally, the backward root takes value in $\{0.8, 0.9, 0.95, 0.975, 0.99, 1\}$, corresponding to columns (1) to (6).

Table 2 gives the levels of the MADs measures for $T = 100$. Unsurprisingly, the forecast accuracy decreases as the forecast horizon h increases[3]: the MAD values obtained for $h = 1$ range from 0.935 to 0.962 whereas the ones found for $h = 10$ lie between 4.48 and 4.61.

Tables 3 to 5 report the percent deviation of the MADs from the one of the best model, for $T = 100$, 200 and 400 respectively. It can be seen from Tables 2 and 3 that the best strategy to choose the forecasting model depends heavily on the size of the backward root ϕ. Indeed, for values up to 0.95, the L strategy outperforms all other ones, all the more so as h increases. For this range of ϕ values, the second best strategy is P at the 1%-level: the percent deviation from the L strategy in terms of MAD ranges from 0.5% for $h = 1$ to 6.9% for $h = 10$ when $T = 100$, from 0% ($h = 1$) to 2.4% ($h = 10$) when $T = 200$ and from 0% ($h = 1$) to 0.6% ($h = 10$) when $T = 400$. For backward root values up to 0.95, the worst strategy is D. For instance, Table 4 indicates that this strategy's percent deviation from the L strategy in terms of MAD ranges from 0.9% for $h = 1$ to 34.7% for $h = 10$ when $T = 200$.

By contrast, when ϕ approaches 1 from below, i.e. local alternatives for the unit root pretest, the D strategy is the best one, the strategy P being the second best again at the 1%-level. These conclusions hold for larger sample size, see Tables 4 and 5, and provide support to Box and Jenkins [1976]'s recommendation. Furthermore, a closer look at Tables 3 to 5 also reveals that the forecast performance deterioration of the L strategy when ϕ approaches 1 (first four lines of columns (4) to (6)) is much weaker than the one of the D strategy when ϕ is well below unity (last four lines of columns (1) to (3)), all the more so as the sample size increases. This finding would support the L strategy. Nevertheless, from a practitioner's point of view, as values of $\phi > 0.95$ are typically found for quarterly and monthly macroeconomic data (see Diebold and Kilian [2000]), the P strategy at the 1%-level seems to be a good compromise. In fact, as stressed earlier, its performance is similar to the one of the L strategy for backward root values far from unity — especially for $T > 100$ — while it deteriorates in general less than the one of the L strategy for backward root values as ϕ approaches 1 from below. Indeed, looking at columns (4) to (6) of e.g. Table 3, it can

[3]This finding is confirmed for $T = 200$ and 400. Moreover, the forecast accuracy increases with the sample size T, as expected. The tables giving the levels of the MAD for these sample sizes are not reported to save space, but are available upon request.

be seen that the percent deviations of the L strategy from the best model lie between 0% ($h = 2$) and 6.7% ($h = 10$) whereas the ones of the $P_{1\%}$ range from 0.2% ($h = 1$) to 2.1% ($h = 10$).

As can be seen from Tables 6 to 7, the same conclusions hold for a smaller value of the forward root, *i.e.* $\psi = 0.3$, or for fatter tails in the residuals Student distribution, namely $\lambda = 3$.

Results obtained from demeaned and detrended data — reported in Tables 8 and 9 — are more mitigated. They also support the choice of the P strategy at the 1%-level, for the same reasons as above, but for horizons $h = 1, 2$ only. Indeed, for longer forecast horizons, the deterioration of its performance is more sizeable than the one of the L strategy. Indeed, in presence of a deterministic component, the P strategy suffers from higher critical values as well as from one or two more parameters to estimate before demeaning or detrending. This could explain why the power of this strategy deteriorates for horizons longer than 2.

7 Concluding remarks

Our paper's goal was to investigate which strategy to adopt in order to choose the best forecasting model — in terms of accuracy — for MAR(1,1) DGPs: always differencing (D), never differencing (L) or unit root pretesting (P).

As a by-product of our analysis, a preliminary simulations study has revealed that for the MAR(1,1) models considered here, *(i)* the importance density of the SIR algorithm should retain a unit root in the candidate model, *(ii)* the median should be the location measure for the simulated predictive distribution and *(iii)* the Lanne et al. [2012] and Gouriéroux and Jasiak [2016] forecasting approaches produce very similar forecasts accuracy.

The main results obtained here from a large simulation study support Box and Jenkins [1976]'s recommendation to use the model in levels unless the root of the process to forecast is close to unity. Moreover, they confirm the ones obtained in favour of the pretesting strategy by e.g. Campbell and Perron [1991], Stock [1996], Stock and Watson [1999] or Diebold and Kilian [2000] for the linear autoregressive class of models. Extending these works by considering a mixed causal-noncausal autoregressive DGP as well as various levels of the unit root pretest strategy, it turns out that from a practitioner's point of view, the P strategy at the 1%-level seems to be a good compromise for MAR(1,1) models. Indeed, it never departs too much from the best model, unlike the L (respectively D) strategy when the DGP becomes very persistent (respectively with little persistence).

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8 Appendix

TABLE 1: PERFORMANCE OF IMPLEMENTATIONS OF FORECASTS.

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
		Gourieroux and Jasiak (2016)								LLS
γ^*		$\tilde{\gamma}$	$\tilde{\gamma}$	$\tilde{\gamma}$	$\tilde{\gamma}$	1	1	1	1	
c		1	4	1	1	1	4	1	1	
λ^*		$\hat{\lambda}$	$\hat{\lambda}$	3	100	$\hat{\lambda}$	$\hat{\lambda}$	3	100	
MAD(1)	Mean	0.935	0.935	0.933	0.934	0.935	0.935	0.934	0.935	0.934
MAD(2)	Mean	1.70	1.70	1.70	1.70	1.70	1.70	1.70	1.70	1.70
MAD(5)	Mean	3.36	3.36	3.36	3.36	3.35	3.36	3.36	3.36	3.36
MAD(10)	Mean	4.48	4.48	4.48	4.48	4.48	4.48	4.48	4.48	4.47
MAD(1)	Mode	0.948	0.962	0.946	0.949	0.952	0.976	0.957	0.960	
MAD(2)	Mode	1.72	1.74	1.72	1.72	1.73	1.77	1.74	1.75	
MAD(5)	Mode	3.39	3.44	3.40	3.40	3.41	3.48	3.43	3.43	
MAD(10)	Mode	4.53	4.61	4.54	4.54	4.55	4.64	4.59	4.58	
MAD(1)	Median	0.936	0.937	0.934	0.936	0.936	0.937	0.936	0.936	0.935
MAD(2)	Median	1.70	1.70	1.70	1.70	1.70	1.71	1.70	1.71	1.70
MAD(5)	Median	3.36	3.36	3.36	3.36	3.36	3.36	3.36	3.36	3.36
MAD(10)	Median	4.48	4.49	4.48	4.48	4.48	4.49	4.48	4.48	4.48
Coverage(1)		89.3	89.2	89.4	89.3	89.1	89.1	89.2	88.9	89.3
Coverage(2)		88.9	88.9	88.9	88.7	88.9	88.6	89.0	88.6	88.9
Coverage(5)		87.6	87.8	87.7	87.5	87.7	87.6	87.6	87.5	87.8
Coverage(10)		87.6	87.5	87.5	87.3	87.5	87.4	87.4	87.3	87.7

NOTE: *The data generating process has* $\phi = 0.9$, $\psi = 0.6$, $\sigma = 1$, $\lambda = 7$ *and* $T = 200$. *The forecasts are based on* $M = N = 10000$ *draws and the results are based on* $B = 10000$ *Monte Carlo replications. Local maxima with* $\hat{\phi} > \hat{\psi}$ *were chosen in 10.75 percent of the cases, while 6.65 percent has still* $\hat{\phi} < \hat{\psi}$.

TABLE 2: FORECASTING RESULTS FOR T=100.

		(1)	(2)	(3)	(4)	(5)	(6)
ϕ		0.8	0.9	0.95	0.975	0.99	1
MAD(1)	L	0.940	0.941	0.942	0.943	0.943	0.945
MAD(2)	L	1.62	1.70	1.75	1.77	1.79	1.80
MAD(5)	L	2.78	3.32	3.66	3.85	3.97	4.07
MAD(10)	L	3.27	4.50	5.48	6.11	6.58	6.96
MAD(1)	$P_{1\%}$	0.945	0.947	0.947	0.945	0.942	0.938
MAD(2)	$P_{1\%}$	1.64	1.72	1.77	1.78	1.78	1.78
MAD(5)	$P_{1\%}$	2.84	3.44	3.77	3.89	3.93	3.94
MAD(10)	$P_{1\%}$	3.41	4.81	5.80	6.24	6.48	6.57
MAD(1)	$P_{5\%}$	0.948	0.948	0.947	0.945	0.942	0.938
MAD(2)	$P_{5\%}$	1.65	1.73	1.77	1.78	1.78	1.78
MAD(5)	$P_{5\%}$	2.91	3.49	3.79	3.89	3.92	3.94
MAD(10)	$P_{5\%}$	3.56	4.98	5.86	6.25	6.46	6.56
MAD(1)	$P_{10\%}$	0.955	0.952	0.948	0.943	0.940	0.937
MAD(2)	$P_{10\%}$	1.68	1.75	1.77	1.77	1.77	1.77
MAD(5)	$P_{10\%}$	3.04	3.56	3.80	3.88	3.92	3.94
MAD(10)	$P_{10\%}$	3.84	5.14	5.91	6.25	6.45	6.54
MAD(1)	D	0.969	0.953	0.945	0.941	0.939	0.936
MAD(2)	D	1.74	1.76	1.77	1.77	1.77	1.77
MAD(5)	D	3.30	3.64	3.81	3.88	3.91	3.92
MAD(10)	D	4.41	5.39	5.98	6.26	6.43	6.52
Unit root test	ERF 1%	42.5	18.7	9.0	4.7	2.8	1.7
Unit root test	ERF 5%	70.4	43.7	23.4	13.2	7.7	4.1
Unit root test	ERF 10%	87.0	64.6	40.3	24.2	14.9	7.9
Local maximum chosen		9.2	14.8	20.4	25.0	29.5	35.9
Maximum with $\hat{\psi} > \hat{\phi}$		24.8	14.7	8.4	4.7	2.7	1.9

NOTE: *The data generating process has* $\psi = 0.6$, $\lambda = 7$, *and* $\sigma = 1$. *The forecasts are based on* $M = N = 10000$ *draws and the results are based on* $B = 10000$ *Monte Carlo replications.*

Table 3: Results for T=100. Deviation from best forecast in percent.

		(1)	(2)	(3)	(4)	(5)	(6)
ϕ		0.8	0.9	0.95	0.975	0.99	1
MAD(1)	L	0.0	0.0	0.0	0.2	0.4	1.0
MAD(2)	L	0.0	0.0	0.0	0.0	1.1	1.7
MAD(5)	L	0.0	0.0	0.0	0.0	1.5	3.8
MAD(10)	L	0.0	0.0	0.0	0.0	2.3	6.7
MAD(1)	$P_{1\%}$	0.5	0.6	0.5	0.4	0.3	0.2
MAD(2)	$P_{1\%}$	1.2	1.2	1.1	0.6	0.6	0.6
MAD(5)	$P_{1\%}$	2.2	3.6	3.0	1.0	0.5	0.5
MAD(10)	$P_{1\%}$	4.3	6.9	5.8	2.1	0.8	0.8
MAD(1)	$P_{5\%}$	0.9	0.7	0.5	0.4	0.3	0.2
MAD(2)	$P_{5\%}$	1.9	1.8	1.1	0.6	0.6	0.6
MAD(5)	$P_{5\%}$	4.7	5.1	3.6	1.0	0.3	0.5
MAD(10)	$P_{5\%}$	8.9	10.7	6.9	2.3	0.5	0.6
MAD(1)	$P_{10\%}$	1.6	1.2	0.6	0.2	0.1	0.1
MAD(2)	$P_{10\%}$	3.7	2.9	1.1	0.0	0.0	0.0
MAD(5)	$P_{10\%}$	9.4	7.2	3.8	0.8	0.3	0.5
MAD(10)	$P_{10\%}$	17.4	14.2	7.8	2.3	0.3	0.3
MAD(1)	D	3.1	1.3	0.3	0.0	0.0	0.0
MAD(2)	D	7.4	3.5	1.1	0.0	0.0	0.0
MAD(5)	D	18.7	9.6	4.1	0.8	0.0	0.0
MAD(10)	D	34.9	19.8	9.1	2.5	0.0	0.0
Unit root test	ERF 1%	42.5	18.8	9.0	4.7	2.8	1.7
Unit root test	ERF 5%	70.4	43.7	23.4	13.2	7.7	4.1
Unit root test	ERF 10%	87.0	64.6	40.3	24.2	14.9	7.9
Local maximum chosen		9.2	14.8	20.4	25.0	29.5	35.9
Maximum with $\hat{\psi} > \hat{\phi}$		24.8	14.7	8.4	4.7	2.7	1.9

Note: *The data generating process has* $\psi = 0.6$, $\lambda = 7$, *and* $\sigma = 1$. *The forecasts are based on* $M = N = 10000$ *draws and the results are based on* $B = 10000$ *Monte Carlo replications.*

TABLE 4: RESULTS FOR T=200. DEVIATION FROM BEST FORECAST IN PERCENT.

		(1)	(2)	(3)	(4)	(5)	(6)
ϕ		0.8	0.9	0.95	0.975	0.99	1
MAD(1)	L	0.0	0.0	0.0	0.0	0.3	0.5
MAD(2)	L	0.0	0.0	0.0	0.0	0.0	1.1
MAD(5)	L	0.0	0.0	0.0	0.0	0.5	2.0
MAD(10)	L	0.0	0.0	0.0	0.0	0.0	3.5
MAD(1)	$P_{1\%}$	0.0	0.1	0.2	0.3	0.3	0.1
MAD(2)	$P_{1\%}$	0.6	0.6	0.6	1.1	0.0	0.6
MAD(5)	$P_{1\%}$	0.0	0.6	1.1	1.3	0.8	0.2
MAD(10)	$P_{1\%}$	0.9	1.1	2.4	3.0	0.9	0.5
MAD(1)	$P_{5\%}$	0.1	0.4	0.5	0.4	0.2	0.1
MAD(2)	$P_{5\%}$	0.6	1.2	1.1	1.1	0.0	0.6
MAD(5)	$P_{5\%}$	0.7	1.8	2.2	1.6	0.3	0.2
MAD(10)	$P_{5\%}$	1.8	3.6	5.3	4.1	0.6	0.2
MAD(1)	$P_{10\%}$	1.0	1.2	1.0	0.5	0.1	0.0
MAD(2)	$P_{10\%}$	1.9	2.4	1.7	1.1	0.0	0.6
MAD(5)	$P_{10\%}$	4.3	5.4	3.8	1.6	0.0	0.0
MAD(10)	$P_{10\%}$	8.6	11.4	9.0	4.6	0.3	0.0
MAD(1)	D	3.6	1.8	0.9	0.3	0.0	0.0
MAD(2)	D	6.8	3.5	2.3	1.1	0.0	0.0
MAD(5)	D	17.7	9.2	4.1	1.3	0.0	0.0
MAD(10)	D	34.7	20.3	10.3	4.5	0.2	0.0
Unit root test	ERF 1%	73.9	42.6	14.0	4.5	1.4	0.5
Unit root test	ERF 5%	94.3	87.2	57.3	26.2	10.1	3.4
Unit root test	ERF 10%	97.7	96.4	84.4	51.8	23.2	8.0
Local maximum chosen		6.6	10.8	14.6	16.9	19.1	24.9
Maximum with $\hat{\psi} > \hat{\phi}$		17.8	6.7	2.3	0.8	0.3	0.2

NOTE: *The data generating process has* $\psi = 0.6$, $\lambda = 7$, *and* $\sigma = 1$. *The forecasts are based on* $M = N = 10000$ *draws and the results are based on* $B = 10000$ *Monte Carlo replications.*

TABLE 5: RESULTS FOR T=400. DEVIATION FROM BEST FORECAST IN PERCENT.

		(1)	(2)	(3)	(4)	(5)	(6)
ϕ		0.8	0.9	0.95	0.975	0.99	1
MAD(1)	L	0.0	0.0	0.0	0.0	0.2	0.3
MAD(2)	L	0.0	0.0	0.0	0.0	0.0	0.6
MAD(5)	L	0.0	0.0	0.0	0.0	0.0	1.0
MAD(10)	L	0.0	0.0	0.0	0.0	0.0	2.0
MAD(1)	$P_{1\%}$	0.0	0.0	0.0	0.1	0.3	0.1
MAD(2)	$P_{1\%}$	0.0	0.0	0.0	0.0	0.6	0.0
MAD(5)	$P_{1\%}$	0.0	0.0	0.0	0.3	0.8	0.3
MAD(10)	$P_{1\%}$	0.6	0.0	0.0	1.0	1.4	0.3
MAD(1)	$P_{5\%}$	0.0	0.0	0.0	0.3	0.2	0.0
MAD(2)	$P_{5\%}$	0.0	0.0	0.0	0.0	0.6	0.0
MAD(5)	$P_{5\%}$	0.4	0.0	0.3	0.8	0.8	0.0
MAD(10)	$P_{5\%}$	0.6	0.2	0.2	2.4	1.6	0.2
MAD(1)	$P_{10\%}$	0.2	0.1	0.3	0.3	0.0	0.0
MAD(2)	$P_{10\%}$	0.0	0.0	0.6	1.2	0.0	0.0
MAD(5)	$P_{10\%}$	0.7	0.6	1.7	2.4	0.8	0.0
MAD(10)	$P_{10\%}$	1.5	0.9	3.8	5.4	1.7	0.0
MAD(1)	D	3.6	1.7	0.8	0.2	0.0	0.0
MAD(2)	D	7.6	4.2	1.8	0.6	0.0	0.0
MAD(5)	D	19.8	10.8	5.4	2.4	0.5	0.0
MAD(10)	D	35.9	22.1	12.0	5.9	1.6	0.0
Unit root test	ERF 1%	96.3	96.9	78.4	27.5	5.0	0.5
Unit root test	ERF 5%	98.8	99.3	98.9	75.4	27.2	4.4
Unit root test	ERF 10%	99.5	99.8	99.8	91.3	47.7	9.4
Local maximum chosen		3.0	4.9	6.0	6.6	7.4	10.4
Maximum with $\hat{\psi} > \hat{\phi}$		9.9	1.9	0.2	0.0	0.1	0.0

NOTE: *The data generating process has* $\psi = 0.6$, $\lambda = 7$, *and* $\sigma = 1$. *The forecasts are based on* $M = N = 10000$ *draws and the results are based on* $B = 10000$ *Monte Carlo replications.*

TABLE 6: RESULTS FOR T=200 AND $\psi = 0.3$. DEVIATION FROM BEST FORECAST IN PERCENT.

		(1)	(2)	(3)	(4)	(5)	(6)
ϕ		0.8	0.9	0.95	0.975	0.99	1
MAD(1)	L	0.0	0.0	0.0	0.0	0.1	0.6
MAD(2)	L	0.0	0.0	0.0	0.0	0.0	1.3
MAD(5)	L	0.0	0.0	0.0	0.0	0.0	2.2
MAD(10)	L	0.0	0.0	0.0	0.0	0.0	3.6
MAD(1)	$P_{1\%}$	0.0	0.0	0.2	0.2	0.1	0.1
MAD(2)	$P_{1\%}$	0.0	0.7	0.7	0.7	0.0	0.0
MAD(5)	$P_{1\%}$	0.0	0.4	0.8	1.5	0.7	0.7
MAD(10)	$P_{1\%}$	0.0	0.4	1.8	2.6	1.0	0.5
MAD(1)	$P_{5\%}$	0.1	0.1	0.5	0.4	0.1	0.1
MAD(2)	$P_{5\%}$	0.0	0.7	1.4	0.7	0.0	0.0
MAD(5)	$P_{5\%}$	0.0	0.4	2.0	1.9	0.4	0.4
MAD(10)	$P_{5\%}$	0.5	0.7	4.1	3.9	1.0	0.2
MAD(1)	$P_{10\%}$	0.2	0.9	1.0	0.4	0.0	0.1
MAD(2)	$P_{10\%}$	0.7	2.1	2.0	1.3	0.0	0.0
MAD(5)	$P_{10\%}$	1.0	3.9	4.3	2.2	0.0	0.4
MAD(10)	$P_{10\%}$	2.4	6.7	8.8	4.5	0.5	0.0
MAD(1)	D	5.5	2.6	1.0	0.2	0.0	0.0
MAD(2)	D	10.2	4.9	2.7	1.3	0.0	0.0
MAD(5)	D	20.5	10.8	5.1	1.9	0.0	0.0
MAD(10)	D	35.2	21.6	10.8	4.5	0.5	0.0
Unit root test	ERF 1%	95.4	78.3	29.0	8.2	2.4	0.8
Unit root test	ERF 5%	98.9	98.8	75.5	36.7	13.7	4.6
Unit root test	ERF 10%	99.5	99.9	91.5	59.6	27.0	9.3
Local maximum chosen		12.4	17.2	20.8	23.8	27.8	36.7
Maximum with $\hat{\psi} > \hat{\phi}$		3.4	0.6	0.2	0.1	0.2	0.2

NOTE: *The data generating process has* $\psi = 0.3$, $\lambda = 7$, *and* $\sigma = 1$. *The forecasts are based on* $M = N = 10000$ *draws and the results are based on* $B = 10000$ *Monte Carlo replications.*

TABLE 7: RESULTS FOR T=200 AND $\lambda = 3$. DEVIATION FROM BEST FORECAST IN PERCENT.

		(1)	(2)	(3)	(4)	(5)	(6)
ϕ		0.8	0.9	0.95	0.975	0.99	1
MAD(1)	L	0.0	0.0	0.0	0.0	0.8	0.0
MAD(2)	L	0.0	0.0	0.0	0.0	0.4	0.4
MAD(5)	L	0.0	0.0	0.0	0.0	0.0	1.9
MAD(10)	L	0.0	0.0	0.0	0.0	0.0	2.9
MAD(1)	$P_{1\%}$	0.0	0.0	0.0	0.8	0.8	0.0
MAD(2)	$P_{1\%}$	0.0	0.0	0.4	0.4	0.4	0.0
MAD(5)	$P_{1\%}$	0.8	0.5	0.4	0.8	0.2	0.4
MAD(10)	$P_{1\%}$	1.6	1.0	0.5	1.2	0.5	0.6
MAD(1)	$P_{5\%}$	0.0	0.0	0.0	0.0	0.8	0.0
MAD(2)	$P_{5\%}$	0.0	0.0	0.4	0.4	0.0	0.0
MAD(5)	$P_{5\%}$	0.8	0.5	0.8	1.0	0.2	0.2
MAD(10)	$P_{5\%}$	1.6	1.0	1.2	1.9	0.7	0.2
MAD(1)	$P_{10\%}$	0.0	0.0	0.0	0.8	0.0	0.0
MAD(2)	$P_{10\%}$	0.0	0.5	0.9	0.4	0.0	0.0
MAD(5)	$P_{10\%}$	0.8	1.1	2.1	2.0	0.2	0.0
MAD(10)	$P_{10\%}$	1.8	2.1	3.5	3.4	0.8	0.0
MAD(1)	D	4.2	1.7	0.8	0.8	0.0	0.0
MAD(2)	D	9.1	4.6	2.2	0.9	0.0	0.0
MAD(5)	D	22.8	12.2	5.6	2.8	0.2	0.0
MAD(10)	D	39.2	21.9	10.4	4.9	0.7	0.0
Unit root test	ERF 1%	97.0	92.0	63.7	28.8	7.9	0.8
Unit root test	ERF 5%	97.9	97.5	89.0	61.3	25.9	4.8
Unit root test	ERF 10%	98.0	98.0	95.2	77.4	42.3	9.9
Local maximum chosen		0.2	0.4	0.6	0.6	0.7	1.0
Maximum with $\hat{\psi} > \hat{\phi}$		2.6	0.5	0.1	0.0	0.0	0.0

NOTE: *The data generating process has* $\psi = 0.6$, $\lambda = 3$, *and* $\sigma = 1$. *The forecasts are based on* $M = N = 10000$ *draws and the results are based on* $B = 10000$ *Monte Carlo replications.*

TABLE 8: RESULTS FOR T=200. DEMEANED DATA. DEVIATION FROM BEST FORECAST IN PERCENT.

		(1)	(2)	(3)	(4)	(5)	(6)
ϕ		0.8	0.9	0.95	0.975	0.99	1
MAD(1)	L	0.0	0.0	0.0	0.0	0.5	1.1
MAD(2)	L	0.0	0.0	0.0	0.0	0.6	1.7
MAD(5)	L	0.0	0.0	0.0	0.0	1.8	4.0
MAD(10)	L	0.0	0.0	0.0	0.0	2.3	5.9
MAD(1)	$P_{1\%}$	0.8	1.2	0.6	0.3	0.2	0.1
MAD(2)	$P_{1\%}$	1.2	1.8	1.1	0.6	0.0	0.6
MAD(5)	$P_{1\%}$	3.9	4.8	3.0	0.5	0.5	0.8
MAD(10)	$P_{1\%}$	6.7	9.1	6.3	2.3	0.6	0.8
MAD(1)	$P_{5\%}$	1.3	1.5	0.9	0.2	0.1	0.0
MAD(2)	$P_{5\%}$	1.9	2.4	1.1	0.6	0.0	0.6
MAD(5)	$P_{5\%}$	5.7	5.7	3.2	0.3	0.3	0.5
MAD(10)	$P_{5\%}$	10.4	11.7	7.4	2.3	0.3	0.5
MAD(1)	$P_{10\%}$	2.3	1.7	0.7	0.1	0.1	0.0
MAD(2)	$P_{10\%}$	3.7	2.9	1.1	0.6	0.0	0.0
MAD(5)	$P_{10\%}$	10.0	7.4	3.5	0.3	0.0	0.3
MAD(10)	$P_{10\%}$	18.0	15.2	7.8	2.1	0.2	0.2
MAD(1)	D	3.8	2.0	0.7	0.1	0.0	0.0
MAD(2)	D	6.8	3.5	1.7	0.6	0.0	0.0
MAD(5)	D	18.2	8.9	3.5	0.0	0.0	0.0
MAD(10)	D	33.8	18.8	8.3	1.9	0.0	0.0
Unit root test	ERF 1%	40.7	12.9	3.5	1.5	1.0	0.9
Unit root test	ERF 5%	65.1	34.3	12.1	5.1	3.0	2.7
Unit root test	ERF 10%	77.7	51.6	23.9	10.9	6.5	5.7
Local maximum chosen		5.5	9.0	13.0	15.6	17.4	17.3
Maximum with $\hat{\psi} > \hat{\phi}$		18.8	8.5	3.7	1.8	1.1	0.8

NOTE: *The data generating process has* $\psi = 0.6$, $\lambda = 7$, *and* $\sigma = 1$. *The forecasts are based on* $M = N = 10000$ *draws and the results are based on* $B = 10000$ *Monte Carlo replications.*

TABLE 9: RESULTS FOR T=200. DETRENDED DATA. DEVIATION FROM BEST FORECAST IN PERCENT.

		(1)	(2)	(3)	(4)	(5)	(6)
ϕ		0.8	0.9	0.95	0.975	0.99	1
MAD(1)	L	0.0	0.0	0.0	0.0	0.6	0.9
MAD(2)	L	0.0	0.0	0.0	0.0	1.1	1.7
MAD(5)	L	0.0	0.0	0.0	0.5	3.0	3.7
MAD(10)	L	0.0	0.0	0.0	1.4	5.4	7.1
MAD(1)	$P_{1\%}$	1.9	1.8	1.0	0.2	0.1	0.1
MAD(2)	$P_{1\%}$	3.1	3.0	1.7	0.0	0.6	0.0
MAD(5)	$P_{1\%}$	7.7	6.6	3.2	0.3	0.2	0.2
MAD(10)	$P_{1\%}$	12.3	11.4	4.7	0.5	0.5	0.3
MAD(1)	$P_{5\%}$	2.6	2.0	1.0	0.2	0.1	0.1
MAD(2)	$P_{5\%}$	3.7	3.6	2.3	0.0	0.6	0.0
MAD(5)	$P_{5\%}$	10.2	7.5	3.5	0.3	0.0	0.0
MAD(10)	$P_{5\%}$	16.0	13.0	5.0	0.2	0.2	0.1
MAD(1)	$P_{10\%}$	3.5	2.3	1.1	0.1	0.0	0.0
MAD(2)	$P_{10\%}$	5.6	4.1	2.3	0.0	0.0	0.0
MAD(5)	$P_{10\%}$	14.2	9.0	3.5	0.0	0.0	0.0
MAD(10)	$P_{10\%}$	22.3	15.3	5.2	0.0	0.0	0.1
MAD(1)	D	4.8	2.6	1.1	0.1	0.0	0.1
MAD(2)	D	8.1	4.7	2.3	0.0	0.6	0.0
MAD(5)	D	20.8	10.2	3.8	0.0	0.0	0.0
MAD(10)	D	32.5	17.1	5.4	0.0	0.0	0.0
Unit root test	ERF 1%	27.2	7.1	2.3	1.4	1.1	1.1
Unit root test	ERF 5%	46.4	18.0	6.4	3.3	2.4	2.3
Unit root test	ERF 10%	57.7	28.5	11.5	6.2	4.4	4.0
Local maximum chosen		4.3	7.3	10.8	12.5	13.9	14.0
Maximum with $\hat{\psi} > \hat{\phi}$		19.5	9.6	5.0	3.3	2.6	2.5

NOTE: *The data generating process has* $\psi = 0.6$, $\lambda = 7$, *and* $\sigma = 1$. *The forecasts are based on* $M = N = 10000$ *draws and the results are based on* $B = 10000$ *Monte Carlo replications.*